1 Relations and orderings

Definition 1.1.

- R is *reflexive* in A if $\forall a \in A, aRa$.
- R is *symmetric* in A if $\forall a, b \in A, aRb \implies bRa$.
- R is *transitive* in A if $\forall a, b, c \in A, aRa \wedge bRc \implies aRc$.
- R is *antisymmetric* if $\forall a, b \in A, aRb \wedge bRa \implies a = b$.
- R is *asymmetric* if $aRb \implies \neg bRa$.
- R is *connex* if $aRy \vee yRa$. □

Definition 1.2.

- Equivalence relation: reflexive, symmetric, transitive.
- Partial ordering: reflexive, antisymmetric, transitive.
- Strict partial ordering: asymmetric, transitive.
- Linear/total ordering: partial order + connex. □

Definition 1.3 (Dense sets)**.** An linearly ordered set $(X, <)$ is *dense* if it has at least two elements and

$$\forall a, b \in X, \exists x \in X \; [a < b \implies a < x < b]$$

□

Definition 1.4 (Relations)**.** A *binary relation* is a set of ordered pairs. Let R be a binary relation, then

- the *domain* of R is defined as $\text{dom}(R) = \{x \mid \exists y, (x, y) \in R\}$
- and the *range* of R is defined as $\text{ran}(R) = \{y \mid \exists y, (x, y) \in R\}$. □

Definition 1.5. Let $\leq$ be a partial ordering on A, and let $B \subseteq A$.

- $b \in B$ is the *least* element of B if $b \leq x$ for all $x \in B$.
- $b \in B$ is the *minimal* element of B is there exists no $x \in B$ such that $x \leq b$ and $x \neq b$.
- $a \in A$ is a *lower bound* of B if $a \leq x$ for all $x \in B$.
- $a \in A$ is called the *infimum* of B if it is the greatest element of the set of all lower bounds of B (greatest lower bound).
- The *greatest* element, *maximal* element, *upper bound*, *supremum* can be defined similarly.

□

Definition 1.6 (Functions)**.** A *function* f is a binary relation such that for every x there is at most one y for which $(x, y) \in f$:

$$(x, y) \in f \wedge (x, z) \in f \implies y = z.$$ □

Theorem 1.1. *Let A and B and A_i be sets and I an indexing set.*

i. $f[\bigcup\{A_i \mid i \in I\}] = \bigcup\{f[A_i] \mid i \in I\}$.

ii. $f[\bigcap\{A_i \mid i \in I\}] \subseteq \bigcap\{f[A_i] \mid i \in I\}$.

iii. $f[A] - f[B] \subseteq F[A - B]$.

Corollary 1.1.1. *Let A and B and A_i be sets and I an indexing set.*

i. $f^{-1}[\bigcup\{A_i \mid i \in I\}] = \bigcup\{f^{-1}[A_i] \mid i \in I\}$.

ii. $f^{-1}[\bigcap\{A_i \mid i \in I\}] = \bigcap\{f^{-1}[A_i] \mid i \in I\}$.

iii. $f^{-1}[A - B] = f^{-1}[A] - f^{-1}[B]$.

2 Cardinals and cardinalities

Definition 2.1. Let A and B be sets. We say that the *cardinality* of A is less than or equal to the cardinality of B if there is an one-to-one mapping of A into B (i.e. injection). We write $|A| \leq |B|$.

We say that A are *equipotent* (same cardinality) if there is an one-to-one from A onto B (i.e. a bijection). We write $|A| = |B|$. □

Theorem 2.1 (Arithmetic laws)**.**

i. $\kappa + \lambda = \lambda + \kappa$ *and* $\kappa \cdot \lambda = \lambda \cdot \kappa$

ii. $\kappa + (\lambda + \mu) = (\kappa + \lambda) + \mu$ *and* $\kappa \cdot (\lambda \cdot \mu) = (\kappa \cdot \lambda) \cdot \mu$

iii. $\kappa \cdot (\lambda + \mu) = \kappa \cdot \lambda + \kappa \cdot \mu$.

iv. $\kappa^{\lambda+\mu} = \kappa^\lambda \cdot \kappa^\mu$.

v. $(\kappa^\lambda)^\mu = \kappa^{\lambda \cdot \mu}$.

vi. $(\kappa \cdot \lambda)^\mu = \kappa^\mu \cdot \lambda^\mu$.

vii. $\kappa + \kappa = 2 \cdot \kappa$.

viii. $\kappa^2 = \kappa \cdot \kappa$.

ix. $\kappa^\kappa \leq 2^{\kappa \cdot \kappa}$.

Theorem 2.2 (Arithmetic properties of $2^{\aleph_0}$)**.**

i. $\aleph_0 \cdot \aleph_0 = \aleph_0$, $2^{\aleph_0} \cdot 2^{\aleph_0} = 2^{\aleph_0}$.

ii. $\forall n \in \mathbb{N} \; \left[n + 2^{\aleph_0} = \aleph_0 + 2^{\aleph_0} = 2^{\aleph_0} + 2^{\aleph_0} = 2^{\aleph_0}\right]$

iii. $\forall n \in \mathbb{N}, n > 0 \; \left[n \cdot 2^{\aleph_0} = \aleph_0 \cdot 2^{\aleph_0} = 2^{\aleph_0} \cdot 2^{\aleph_0} = 2^{\aleph_0}\right]$

iv. $\forall n \in \mathbb{N}, n > 0 \; \left[\left(2^{\aleph_0}\right)^n = \left(2^{\aleph_0}\right)^{\aleph_0} = 2^{\aleph_0}\right]$

v. $\forall n \in \mathbb{N}, n > 1 \; \left[n^{\aleph_0} = {\aleph_0}^{\aleph_0} = 2^{\aleph_0}\right]$

Theorem 2.3. *The set of all open subsets of the reals has cardinality $2^{\aleph_0}$.*

Theorem 2.4. *Every open interval in the reals has cardinality $2^{\aleph_0}$. Every non-empty open set of reals has cardinality $2^{\aleph_0}$.*

3 Ordinals

Definition 3.1 (Initial segments)**.** Let $(L, <)$ be a linearly ordered set. A set $S \subsetneq L$ is called an *initial segment* if it is closed downwards, i.e. $\forall a \in S, x < a \implies x \in S$. □

Lemma 3.1. *If $(W, <)$ is a well-ordered set and if $f : W \to W$ is an increasing (i.e. order preserving) function, then $f(x) \geq x$ for all $x \in W$.*

Corollary 3.1.1.

i. No well-ordered set is isomorphic to an initial segment of itself.

ii. Each well-ordered set has only one automorphism, the identity.

iii. If W_1 and W_2 are isomorphic well-ordered sets, then the isomorphism between W_1 and W_2 is unique.

Theorem 3.2. *If W_1 and W_2 are well-ordered sets, then one and exactly one of the following holds:*

i. W_1 and W_2 are isomorphic, or

ii. W_2 is isomorphic to an initial segment of W_1, or

iii. W_1 is isomorphic to an initial segment of W_2.

In all cases the isomorphism is unique.

Definition 3.2 (Transitive sets)**.** A set T is *transitive* if every element of T is also a subset of T. In other words, $u \in v \in T \implies u \in T$. □

Definition 3.3 (Ordinals)**.** A set α is an *ordinal* or *ordinal number* if α if transitive and α is well-ordered by $\in_\alpha$. □

Definition 3.4 (Successor ordinals)**.** The successor of ordinal α is given by $\alpha \cup \{\alpha\}$. If $\alpha = \beta + 1$ for some ordinal β then it is called a *successor ordinal*. Otherwise it is called a *limit ordinal*. □

Definition 3.5 (Supremum)**.** Define $\sup S = \bigcup S$. □

Theorem 3.3. *Let α be an ordinal.*

i. $\alpha \notin \alpha$.

ii. Every element of α is an ordinal.

iii. If β is an ordinal, $\alpha \subset \beta \iff \alpha \in \beta$.

Theorem 3.4. *Let α, β, γ be ordinal numbers.*

i. If $\alpha < \beta$ and $\beta < \gamma$, then $\alpha < \gamma$.

ii. $\alpha < \beta$ and $\beta < \alpha$ cannot both be true.

iii. $\alpha < \beta$ or $\alpha = \beta$ or $\beta < \alpha$.

iv. Every set of ordinals is well-ordered by $<$.

v. For every set of ordinal numbers X, there is an ordinal number $\alpha \notin X$, namely $\bigcup X$.

4 Transfinite recursion and induction

Theorem 4.1 (Transfinite induction)**.** *Let $P(x)$ be a property. Assume that*

- *$P(0)$ holds.*
- $\forall\alpha\ [P(\alpha) \implies P(\alpha+1)]$.
- *For all limit ordinals $\alpha \neq 0$, if $P(\beta)$ holds for all $\beta < \alpha$, then $P(\alpha)$ holds.*

Then $P(\alpha)$ holds for all ordinals α.

Theorem 4.2 (Parametric transfinite recursion)**.** *Let G_1, G_2, and G_3 be operations defined on the class of all sets. Then there is an unique operation $F(z,\alpha)$ defined on the class of all sets z and all ordinals α such that*

- $F(z,0) = G_1(z)$,
- $F(z,\alpha+1) = G_2(z,\alpha,F(z,\alpha))$,
- $F(z,\alpha) = G_3(z,\{(\beta,F(z,\beta)) \mid \beta < \alpha\})$ *if $\alpha \neq 0$ is a limit ordinal.*

Definition 4.1 (Addition of ordinals)**.** For all ordinals α:

- $\alpha + 0 = \beta$,
- $\alpha + (\alpha + 1) = (\alpha + \beta) + 1$,
- $\alpha + \beta = \sup\{\alpha + \gamma \mid \gamma < \beta\}$ for all limit ordinals $\alpha \neq 0$

□

Addition is left cancellative, associative, left subtractive (exists unique solution to $\alpha = \gamma + \beta$).

Definition 4.2 (Ordinal multiplication)**.** For all ordinals α,

- $\alpha \cdot 0 = 0$.
- $\alpha \cdot (\beta + 1) = (\alpha \cdot \beta) + \alpha$.
- $\alpha \cdot \beta = \sup\{\alpha \cdot \gamma \mid \gamma < \alpha\}$ for all limit ordinals $\beta \neq 0$.

□

Multiplication is left cancellative (not shown), associative, left distributive.

Definition 4.3 (Ordinal exponentiation)**.** For all ordinals α,

- $\alpha^0 = 1$.
- $\alpha^{\beta+1} = \alpha^\beta \cdot \alpha$.
- $\alpha^\beta = \sup\{\alpha^\gamma \mid \gamma < \beta\}$ for all limit ordinals $\beta \neq 0$.

□

5 Alephs

Definition 5.1 (Initial ordinal)**.** An ordinal α is called an *initial ordinal* if it is not equipotent (equinumerous) to any $\beta < \alpha$. □

Definition 5.2 (Hartogs number)**.** For any set A, let $h(A)$ be the least ordinal which is not equipotent to any subset of A. We call $h(A)$ the *Hartogs number* of A. In other words, $h(A)$ is the least ordinal such that $|h(A)| \not\leq |A|$. □

Definition 5.3. Define

- $\omega_0 = \omega$.
- $\omega_{\alpha+1} = h(\omega_\alpha)$.
- $\omega_\alpha = \sup\{\omega_\beta \mid \beta < \alpha\}$. If $\alpha \neq 0$ is a limit ordinal.

□

6 Axiom of choice

Definition 6.1 (Choice functions)**.** Let S be a system of sets. A function g defined on S is called a *choice function* for S if $g(X) \in X$ for all non-empty $X \in S$. □

Axiom of choice There exists a choice function for every system of sets.

Theorem 6.1 (Zorn's lemma)**.** *If every chain in a partially ordered set has an upper bound, then the partially ordered set has a maximal element.*

7 Other things

The axiom schema of replacement Let $P(x,y)$ be a property such that $\forall x, \exists! y\ P(x,y)$. Then for every set A, there is a set B, such that for every $x \in A$, there is a $y \in B$ for which $P(x,y)$ holds.

Definition 7.1. A Dedekind *cut* in $\mathbb{Q}$ is a subset $A \subseteq \mathbb{Q}$ such that

- $A \neq \emptyset$ and $A \neq \mathbb{Q}$.
- $\forall p \in \mathbb{Q}, \forall q \in A\ (p < q \implies p \in A)$.
- A does not have a greatest element. □

Definition 7.2 (Open sets)**.** A set $A \subseteq \mathbb{R}$ is open if

$$\forall a \in A, \exists \delta > 0\ (|x - a| < \delta \implies x \in A).$$

In other words, there is an open interval (neighbourhood) $(a-\delta, a+\delta) \subseteq A$.

A set B is *closed* if $\mathbb{R} \setminus B$ is open. □

Theorem 7.1. *Every system of mutually disjoint open intervals in $\mathbb{R}$ is at most countable.*

□

Definition 7.3 (Accumulation points)**.** $a \in \mathbb{R}$ is an *accumulation (limit) point* of $A \subseteq \mathbb{R}$ if

$$\forall \delta > 0, \exists x \in A\ (x \neq a\ \wedge |x - a| < \delta).$$

□

Definition 7.4 (Isolated points)**.** $a \in \mathbb{R}$ is an *isolated point* of $A \subseteq \mathbb{R}$ if

$$\exists \delta > 0, \forall x \neq a\ (|x - a| < \delta \implies x \notin A).$$

□

Definition 7.5 (Perfect sets)**.** A non-empty set A is called a *perfect set* if A is closed without isolated points. □

$\mathbb{Z}$:

$$(a,b) \sim (c,d) \iff a + d = b + c.$$
$$[(a,b)] < [(c,d)] \iff a + d <_{\mathbb{N}} b + c.$$
$$[(a,b)] + [(c,d)] = [(a+c, b+d)]$$
$$[(a,b)] \cdot [(c,d)] = [(ac+bd, ad+bc)]$$

$\mathbb{Q}$:

$$(a,b) \sim (c,d) \iff ad = bc.$$
$$[(a,b)] < [(c,d)] \iff ad <_{\mathbb{N}} bc$$
$$[(a,b)] + [(c,d)] = [(ad+bc, bd)]$$
$$[(a,b)] \cdot [(c,d)] = [(ac, bd)]$$