MA3211

Complex Analysis I

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1 Complex numbers

This section is a quick review of complex numbers.

Definition 1.1 (Complex numbers)**.** A *complex number* takes the form $z = x + iy$, where $x, y \in \mathbb{R}$ and $i^2 = -1$. Furthermore we define the *real part* of z as $\Re(z) = x$ and the *imaginary part* of z as $\Im(z) = y$. □

Two complex numbers are equal iff both their real and imaginary parts are equal. The set of complex numbers $\mathbb{C}$ forms a field with the following operations:

Definition 1.2 (Operations in $\mathbb{C}$)**.**

$$\begin{aligned}(x_1 + iy_1) + (x_2 + iy_2) &= (x_1 + x_2) + i(y_1 + y_2)\\ (x_1 + iy_1) \cdot (x_2 + iy_2) &= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1).\end{aligned}$$ □

Division is more easily performed if we multiply the numerator and denominator with a constant that makes the denominator real. We will see how this is done later.

It is also possible to identify a complex number with a vector in $\mathbb{R}^2$. Then we have the familiar notion of length.

Definition 1.3 (Modulus)**.** Define the modulus of a complex number $z = x + iy$ as

$$|z| = \sqrt{x^2 + y^2}.$$ □

The distance between two complex numbers z_1 and z_2 is given through the same way for vectors: $|z_1 - z_2|$. We also have the following relations regarding the modulus:

$$\begin{aligned}\Re(z) \le |\Re(z)| &\le |z|\\ \Im(z) \le |\Im(z)| &\le |z|\\ |z_1z_2| &= |z_1||z_2|\end{aligned}$$

Definition 1.4 (Conjugate)**.** The *conjugate* of $z = x + iy$ is given by $\bar{z} = x - iy$. □

The following are some simple properties of the complex conjugate:

- $\mathfrak{R}(z) = (z + \overline{z})/2$, $\mathfrak{I}(z) = (z - \overline{z})/2$.
- $\overline{z_1 \pm z_2} = \overline{z_1} \pm \overline{z_2}$.
- $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$, $\overline{z_1/z_2} = \overline{z_1}/\overline{z_2}$.
- $z\overline{z} = |z|^2$.

The last property is also what we can use to perform division easily:

$$\frac{z_1}{z_2} = \frac{z_1}{z_2}\frac{\overline{z_2}}{\overline{z_2}} = \frac{z_1\overline{z_2}}{|z_2|^2}.$$

Theorem 1.1 (Triangle inequality). *For $z_1, z_2 \in \mathbb{C}$, we have*

$$|z_1 + z_2| \le |z_1| + |z_2|.$$

Generally,

$$|z_1 + z_2 + \cdots + z_n| \le |z_1| + |z_2| + \cdots + |z_n|.$$

Proof. We have

$$\begin{aligned} |z_1 + z_2|^2 &= (z_1 + z_2)\overline{(z_1 + z_2)} \\ &= |z_1|^2 + 2\mathfrak{R}(z_1\overline{z_2}) + |z_2|^2 \\ &\le |z_1|^2 + 2z_1\overline{z_2} + |z_2|^2 \\ &= (|z_1| + |z_2|)^2. \end{aligned}$$

We can show the general case with induction. ■

Corollary 1.1.1 (Reverse triangle inequality). *For $z_1, z_2 \in \mathbb{C}$, we have*

$$||z_1| - |z_2|| \le |z_1 - z_2|.$$

Proof. Using the triangle inequality we have $|z_1| = |z_1 - z_2 + z_2| \le |z_1 - z_2| + |z_2|$, so we have $|z_1| - |z_2| \le |z_1 - z_2|$. Repeating the same for the roles reversed, we also have $|z_2| - |z_1| \le |z_2 - z_1| = |z_1 - z_2|$. Notice that $|z_2| - |z_1| = -(|z_1| - |z_2|)$, so in both cases $||z_1| - |z_2|| \le |z_1 - z_2|$. ■

Taking the vector analogy further, every non-zero complex number has a polar form representation:

Definition 1.5 (Polar form). The *polar form* of a complex number $z = x + iy$ is given by $z = r(cos\theta + i\sin\theta)$ with $r = |z|$ and $\theta = \arctan y/x$. □

Since the trigonometric functions are periodic, there can be multiple values of θ that represent z.

Definition 1.6 (Argument). The set of all possible θs is called the *argument* of z, denoted as $\arg z$. In other words

$$\arg z = \{\theta \mid z = r(\cos\theta + i\sin\theta)\}.$$

If $\Theta \in \arg z$ and $-\pi < \Theta \le \pi$, we call Θ the *principle argument* of z and write $\operatorname{Arg} z = \Theta$. □

Theorem 1.2 (Euler's formula).

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

This also means that for any complex number z we can write $z = re^{i\theta}$. Also note that $\overline{z} = re^{-i\theta}$.

Theorem 1.3 (de Moivre's Theorem). *For $n \in \mathbb{Z}$,*

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

Proof. Inductively show $(e^{i\theta})^n = e^{i\theta n}$, result follows. ∎

Since we can find exponents, naturally we are interested in roots.

Definition 1.7 (Roots). For some $z_0 \in \mathbb{C}$, The solutions z that satisfy $z^n = z_0$ are called the *n-th roots* of z_0. □

If $z = re^{i\theta}$ is a n-th root, then $z^n = r^n e^{in\theta} = z_0 = r_0 e^{i\theta_0}$, giving us the relations

$$r^n = r_0 \qquad \theta = \frac{\theta_0 + 2k\pi}{n}, k = 0, 1, \dots, n-1.$$

The following are some definitions in topology.

Definition 1.8 (Open balls). An *open ball* centred at z_0 with radius r is the set of points

$$B(z_0, r) = \{z \in \mathbb{C} \mid |z - z_0| < r\}.$$ □

Definition 1.9 (Interior points, exterior points, boundary points). Let $S \subseteq \mathbb{C}$ and $z \in \mathbb{C}$.

- z is an *interior point* of S if there is an open ball $B(z, r) \subseteq S$.
- z is an *exterior point* of S if there is an open ball $B(z, r) \cap S = \emptyset$.
- z is a *boundary point* of S if for all $r > 0$, $B(z, r) \cap S \neq \emptyset$ and $B(z, r) \cap S^c \neq \emptyset$ [1]

□

Definition 1.10 (Boundary of a set). The *boundary* of $S \subseteq \mathbb{C}$, denoted ∂S, is the set of all boundary points of S. □

Definition 1.11 (Open sets, closed sets). A set $S \subseteq C$ is called *open* if $\partial S \cap S \neq \emptyset$, i.e. S does not contain any of its boundary points. A set $S \subseteq C$ is called *closed* if $\partial S \subseteq S$, i.e. S contains its boundary points. □

Note that a set can be both not open and not closed, in other words, a set that is not open might not be closed!

Theorem 1.4. *$S \subseteq \mathbb{C}$ is open iff S^c is closed.*

Definition 1.12 (Closure). The closure of $S \subseteq \mathbb{C}$ is the set $\overline{S} = S \cup \partial S$. □

[1] $S^c = \mathbb{C} - S$ is the complement of S.

Definition 1.13 (Closed segments). Let $z_1, z_2 \in \mathbb{C}$. The line segment joining them is denoted

$$[z_1, z_2] = \{z \in \mathbb{C} \mid z = z_1 + t(z_2 - z_1), 0 \leq t \leq 1\}.$$

A polygonal line is a finite union of line segments. □

Definition 1.14 (Connected sets, domains). An open set $S \subseteq \mathbb{C}$ is called *connected* if any two points $z_1, z_2 \in S$ can be joined by a polygonal line which lies entirely in S. An open connected set is called a domain. □

Example 1.1. All open balls are domains. ◊

Definition 1.15 (Bounded sets). A set $S \subseteq \mathbb{C}$ is *bounded* if there exists $R > 0$ such that for all $z \in S$, $|z| < R$, or equivalently $S \subseteq B(0, R)$. A set that is not bounded is called *unbounded*. □

Definition 1.16 (Compact sets). A set that is closed and bounded is called *compact*. □

Example 1.2. All closed balls are compact. ◊

2 Analytic functions

Definition 2.1 (Complex functions). Let $S \subseteq \mathbb{C}$. A function $f : S \to \mathbb{C}$ is called a complex valued function of a complex variable. □

A complex function may be thought of as two real valued functions of real variables:

$$f(x + iy) = u(x, y) + iv(x, y).$$

2.1 Limits

Definition 2.2 (Limits). Let f be a complex function defined in some deleted open ball $B(z_0, r) - \{z_0\}$ of z_0. We say w_0 is the *limit* of f as z approaches z_0, and write

$$\lim_{z \to z_0} f(z) = w_0$$

if

$$\forall \epsilon > 0, \exists \delta > 0, \ [|z - z_0| < \delta \implies |f(z) - w_0| < \epsilon]$$

or in other words

$$z \in B(z_0, \delta) - \{z_0\} \implies f(z) \in B(w_0, \epsilon).$$

□

Example 2.1. Let $f(z) = z^2$. Prove that $\lim_{z \to i} f(z) = -1$. Let $\epsilon > 0$. Choose $\delta = \min(1, \epsilon/3)$. Then when $|z - i| < \delta \leq 1$,

$$\begin{aligned} |z + i| &= |z - i + 2i| \\ &\leq |z - i| + |2i| \\ &\leq 1 + 2 \end{aligned}$$

Thus

$$|z-i|<\delta \implies |z-i||z+i|<\frac{\epsilon}{3}\cdot 3$$
$$\implies \left|z^2-(-1)\right|<\epsilon.$$

$\diamondsuit$

Theorem 2.1. *Let $z_0 = x_0 + iy_0$, $w_0 = u_0 + iv_0$ and let $f(z) = u(x, y) + iv(x, y)$. Then $\lim_{z\to z_0} f(z) = w_0$ iff $\lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0$ and $\lim_{(x,y)\to(x_0,y_0)} v(x, y) = v_0$.*

Proof. We have the following:

$$|u(x,y)-u_0| = |\Re(f(z)-w_0)| \le |f(z)-w_0|$$
$$|v(x,y)-v_0| = |\Im(f(z)-w_0)| \le |f(z)-w_0|$$

$\blacksquare$

Theorem 2.2. *Suppose that $\lim_{z\to z_0} f(z) = A$ and $\lim_{z\to z_0} g(z) = B$. Then*

(i) $\lim_{z\to z_0}[f(z) \pm g(z)] = A \pm B$.

(ii) $\lim_{z\to z_0} f(z)g(z) = AB$.

(iii) *If $B \neq 0$, then $\lim_{z\to z_0} f(z)/g(z) = A/B$.*

Definition 2.3 (Limits with infinity)**.** The statement $\lim_{z\to\infty} f(z) = w$ means $\lim_{z\to 0} f(1/z) = w$. The statement $\lim_{z\to z_0} f(x) = \infty$ means that $\lim_{z\to z_0} 1/f(z) = 0$. $\square$

2.2 Continuity

Definition 2.4 (Continuity)**.** The function f is said to be *continuous* at z_0 if $\lim_{z\to z_0} f(z) = f(z_0)$. That is,

$$\forall\epsilon>0, \exists\delta>0\ [|z-z_0|<\delta \implies |f(z)-f(z_0)|<\epsilon]$$

We say that f is continuous in a set S if f is continuous at every point in S. $\square$

2.3 Derivatives

Definition 2.5 (Derivatives)**.** Let f be defined on $B(z_0, r)$ for some $r > 0$. The *derivative* of f at z_0 is defined as

$$\left.\frac{\mathrm{d}}{\mathrm{d}z}f(z)\right|_{z=z_0} = \lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}$$

We also write the derivative as $f'(z_0)$. If $f'(z_0)$ exists, we say that f is *differentiable* at z_0. $\square$

Theorem 2.3 (L'Hopital's rule)**.** *Let f and z be differentiable at z_0. Suppose that $f(z_0) = g(z_0) = 0$ and $g'(z_0) \neq 0$. Then*

$$\lim_{z\to z_0}\frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}$$

The standard rules of differentiation apply such as the chain rule or product rule. We will assume they are known and not write them down.

Definition 2.6 (Partial derivatives)**.** The *partial derivative* of a multi variable function f at a point (x_0, y_0) is defined as

$$\frac{\partial}{\partial x} f(x_0, y_0) = \lim_{x \to x_0} \frac{h(x, y_0) - h(x_0, y_0)}{x - x_0}$$

Of course this is easily generalised to a higher arity. We also write $f_x = \frac{\partial f}{\partial x}$. □

Let $f(z) = u(x, y) + iv(x, y)$. If $f'(z_0)$ exists, where $z_0 = x_0 + iy_0$, then we must obtain the same limit no matter which path we take.

Taking the path along the line $y = y_0$,

$$\begin{aligned} f'(z_0) &= \lim_{(x,y_0)\to(x_0,y_0)} \frac{u(x, y_0) - u(x_0, y_0) + i[v(x, y_0) - v(x_0, y_0)]}{(x - x_0) + i(y_0 - y_0)} \\ &= \lim_{x \to x_0} \frac{u(x, y_0) - u(x_0, y_0)}{x - x_0} + i \lim_{x \to x_0} \frac{v(x, y_0) - v(x_0, y_0)}{x - x_0} \\ &= \frac{\partial}{\partial x} u(x_0, y_0) + i \frac{\partial}{\partial y} v(x_0, y_0). \end{aligned}$$

Taking the path along the line $x = x_0$,

$$\begin{aligned} f'(z_0) &= \lim_{(x,y_0)\to(x_0,y_0)} \frac{u(x_0, y) - u(x_0, y_0) + i[v(x_0, y) - v(x_0, y_0)]}{(x_0 - x_0) + i(y - y_0)} \\ &= \frac{1}{i} \lim_{y \to y_0} \frac{u(x_0, y) - u(x_0, y_0)}{y - y_0} + \lim_{y \to y_0} \frac{v(x_0, y) - v(x_0, y_0)}{y - y_0} \\ &= -i \frac{\partial}{\partial x} u(x_0, y_0) + \frac{\partial}{\partial y} v(x_0, y_0). \end{aligned}$$

Theorem 2.4 (Cauchy Riemann equations)**.** *$f(z) = u(x, y) + iv(x, y)$ is differentiable at $z_0 = x_0 + iy_0$ iff u and v must satisfy the following equations:*

$$\begin{cases} \frac{\partial}{\partial x} u(x_0, y_0) = \frac{\partial}{\partial y} v(x_0, y_0) \\ \frac{\partial}{\partial x} v(x_0, y_0) = -\frac{\partial}{\partial y} u(x_0, y_0) \end{cases}.$$

This also means

$$f'(z_0) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}.$$

Theorem 2.5. *Let $f(z) = u(x, y) + iv(x, y)$ be defined in a neighbourhood $B(z_0, \epsilon)$ of the point $z_0 = x_0 + iy_0$. Suppose that the first order partial derivatives of u and v exist in $B(z_0, \epsilon)$ and satisfy the following:*

(i) the satisfy the Cauchy-Riemann equations, and

(ii) they are continuous at (x_0, y_0).

Then f is differentiable at z_0.

Proof. For $z = x + iy \in B(z_0, \epsilon)$ such that $z \neq z_0$, we have

$$\begin{aligned}
&f(z) - f(z_0) \\
&= u(x, y) - u(x, y_0) + u(x, y_0) - u(x_0, y_0) + i[v(x, y) - v(x, y_0) + v(x, y_0) - v(x_0, y_0)]
\end{aligned}$$

By the mean value theorem,

$$\frac{u(a) - u(b)}{y - y_0} = \frac{\partial}{\partial y} u(x, y_1)$$

for some y_1 between y and y_0. Thus

$$u(x, y) - u(x, y_0) = (y - y_0) \frac{\partial}{\partial y} u(x, y_1).$$

Let

$$\epsilon_1 = \frac{\partial}{\partial y} u(x, y_1) - \frac{\partial}{\partial y} u(x_0, y_0)$$

such that

$$\frac{\partial}{\partial y} u(x, y_1) = \frac{\partial}{\partial y} u(x_0, y_0) + \epsilon_1.$$

Since $\partial u / \partial y$ is continuous at (x_0, y_0), $\lim_{(x, y_1) \to (x_0, y_0)} \epsilon_1 = 0$.

Do the same for the other three pairs of terms. If we put it all together, we get

$$\begin{aligned}
&f(z) - f(z_0) \\
&= (y - y_0) \left[\frac{\partial}{\partial y} u(x_0, y_0) + \epsilon_1 \right] + (x - x_0) \left[\frac{\partial}{\partial x} u(x_0, y_0) + \epsilon_2 \right] \\
&+ i(y - y_0) \left[\frac{\partial}{\partial y} v(x_0, y_0) + \epsilon_3 \right] + i(x - x_0) \left[\frac{\partial}{\partial x} v(x_0, y_0) + \epsilon_4 \right] \\
&= \frac{\partial}{\partial x} u(x_0, y_0)(z - z_0) + i \frac{\partial}{\partial x} v(x_0, y_0)(z - z_0) + (\epsilon_2 + i\epsilon_4)(x - x_0) + (\epsilon_1 + i\epsilon_3)(y - y_0),
\end{aligned}$$

such that

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{\partial}{\partial x} u(x_0, y_0) + i \frac{\partial}{\partial x} v(x_0, y_0) + \underbrace{(\epsilon_2 + i\epsilon_4) \frac{x - x_0}{z - z_0} + (\epsilon_1 + i\epsilon_3) \frac{y - y_0}{z - z_0}}_{R}.$$

Note that the trailing term R tends to 0 as $z \to z_0$:

$$\begin{aligned}
R &\leq (|\epsilon_2| + |\epsilon_4|) \left| \frac{x - x_0}{z - z_0} \right| + (|\epsilon_1| + |\epsilon_3|) \left| \frac{y - y_0}{z - z_0} \right| \\
&\leq |\epsilon_1| + |\epsilon_2| + |\epsilon_3| + |\epsilon_4|.
\end{aligned}$$

Thus the derivative exists and is given by

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \frac{\partial}{\partial x} u(x_0, y_0) + i \frac{\partial}{\partial x} v(x_0, y_0).$$

■

Example 2.2. Let $f(z) = x^3 + i(1-y)^3$. We want to find the set on which f is differentiable.

The first order partial derivatives:

$$\frac{\partial u}{\partial v} = 3x^2 \qquad \frac{\partial u}{\partial y} = 0 \qquad \frac{\partial v}{\partial x} = 0 \qquad \frac{\partial v}{\partial y} = -3(1-y)^2.$$

Solve the Cauchy-Riemann equations:

$$\begin{cases} 3x^2 = -3(1-y)^2 \\ 0 = 0 \end{cases}.$$

The only solution is at $x = 0$ and $y = 1$. The first order partial derivatives of u and v are continuous everywhere but since the Cauchy-Riemann equations are only satisfied at $z = i$, thus we conclude that f is differentiable only at $z = i$, and

$$f'(i) = 0.$$

◊

2.4 Analytic functions

Definition 2.7 (Analytic functions)**.** Let S be a set. A function f is said to be *analytic* in S if

(i) S is an open set and $f'(z)$ exists for all $z \in S$, or

(ii) if f is analytic in an open set containing S

We say f is analytic at a point z_0 if f is analytic in some open ball $B(z_0, r)$. □

Definition 2.8 (Entire functions)**.** If f is analytic in $\mathbb{C}$, then we call f an *entire function.* □

Example 2.3. We have seen previously that $f(z) = x^3 + i(1-y)^3$ is differentiable only at $z = i$. However at all other points in $B(0, r)$, it is not differentiable. Thus f is nowhere analytic. ◊

The previous example should make it quite clear that if a function is differentiable at finitely many points, then it is nowhere analytic.

Theorem 2.6. *If f is analytic in a domain D and if $f'(z) = 0$ everywhere in D, then $f(z)$ is constant in D.*

Proof. Let $f(z) = u(x, y) + iv(x, y)$. Then

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = 0.$$

It follows that $\partial u/\partial x = \partial u/\partial y = 0$ and $\partial v/\partial x = \partial v/\partial y = 0$ on D. Hence u and v are constants. ■

Theorem 2.7. *Let $f(z)$ be a function that is analytic in D. Each of the following conditions alone imply that f is constant in D.*

(i) $\Re f(z)$ is constant in D.

(ii) $f(z)$ *is real valued for all* $z \in D$*.*

(iii) $\overline{f(z)}$ *is analytic in* D*.*

(iv) $|f(z)|$ *is constant in* D*.*

(v) $\operatorname{Arg} f(z)$ *is constant in* D*.*

Proof. Let $f(z) = u(x, y) + iv(x, y)$.

(i) Then u is constant and so the derivatives of u are all 0. Then from the Cauchy-Riemann equations $\frac{\partial v}{\partial x} = 0$ as well. Thus $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 0$.

(ii) If f is real valued then $v = 0$, and so similar to the above, the derivatives end up being 0.

(iii) If $f(z) = u(z) + iv(z)$ and $\overline{f(z)} = u(z) - iv(z)$ are both analytic in D, then they satisfy the Cauchy-Riemann equations:

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} & \qquad \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial x} &= -\frac{\partial v}{\partial y} & \qquad \frac{\partial u}{\partial y} &= \frac{\partial v}{\partial x}. \end{aligned}$$

Solving this set of equations show us that the partial derivatives are all 0.

(iv) Let $|f(z)| = c$ where c is a constant. Then $|f(z)|^2 = f(z)\overline{f(z)} = c^2$. If $r = 0$ then $f(z) = 0$ is a constant. Otherwise, $\overline{f(z)}$ is analytic on D by the quotient rule.

(v) If $\operatorname{Arg} f(z) = c$ is constant, then the ratio $v(z)/u(z) = \arctan c = d$ is also a constant. Then by the Cauchy-Riemann equations:

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} & \qquad \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial x} &= d\frac{\partial u}{\partial y} & \qquad \frac{\partial u}{\partial y} &= -d\frac{\partial u}{\partial x}. \end{aligned}$$

Solving this set of equations show us that the partial derivatives are all 0.

■

2.5 Harmonic functions

Definition 2.9 (Harmonic functions)**.** Let S be a set. A function $f : S \to \mathbb{R}$ is said to be *harmonic* in S if

(i) f has continuous first and second partial derivatives, and

(ii) f satisfies the Laplace equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

□

Theorem 2.8. *If* $f(z) = u(x, y) + iv(x, y)$ *is analytic in a domain* D*, then* u *and* v *are harmonic in* D*. We call* v *a* harmonic conjugate *of* u *in* D*.*

Proof. Since f is differentiable in D, it satisfies the Cauchy-Riemann equations. Differentiating these equations once more gives

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \qquad\qquad \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y}.$$

This means

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Thus u is harmonic. We can do the same for v to see that it too is harmonic. ■

Example 2.4. Given $u(x, y) = y^3 - 3x^2 y$, we want to find all of its harmonic conjugates.

Firstly, $\partial u/\partial x = -6xy$ and $\partial u/\partial y = 3y^2 - 3x^2$. Solve the Cauchy-Riemann equations:

$$\frac{\partial v}{\partial y} = -6xy \qquad\qquad \frac{\partial v}{\partial x} - -3y^2 + 3x^2$$

The end result is $v(x, y) = -3xy^2 + x^3 + C$. ◊

3 Elementary functions

We want to construct some common complex analytic functions. We will be looking at their properties as real functions, and extending them to the complex plane.

3.1 Exponential function

The main properties of the exponential function are

$$f(x + i0) = e^x \qquad\qquad f'(z) = f(z).$$

It can be checked that

$$f(x + iy) = e^x(\cos y + i \sin y)$$

satisfies the properties. It satisfies the Cauchy-Riemann equations everywhere and is entire. Thus

Definition 3.1 (Exponential function)**.** Define for all $z = x + iy \in \mathbb{C}$, the exponential function

$$e^z = \exp(z) = e^x(\cos y + i \sin y).$$

□

Notice that for the case where $\theta \in \mathbb{R}$,

$$e^{i\theta} = \cos\theta + i \sin\theta$$

which is Euler's formula. Thus we can also write

$$e^{x+iy} = e^x e^{iy}.$$

Note that $|e^{iy}| = 1$. Therefore,

$$|e^{x+iy}| = e^x.$$

3.2 Logarithm function

We want to define an inverse for the exponential function. One problem is that the complex exponential is not even one-to-one. This is because $\exp(z) = \exp(z + 2n\pi i)$ for all $n \in \mathbb{Z}$.

Theorem 3.1. *The range of the complex exponential function is* $\mathbb{C} \setminus \{0\}$.

Proof. Take any $w = r_0 e^{i\theta_0} \neq 0$. We show that there exists a $z = x + iy$ such that $e^z = w$, or in other words $e^x e^{iy} = r_0 e^{i\theta_0}$. Simply by solving the previous equation, we have solutions $z = \ln r + i(\theta_0 + 2n\pi)$ for all $n \in \mathbb{Z}$. ∎

The above theorem also clearly gives us a definition for the logarithm function:

$$\log z = \{\ln|z| + i\theta \mid \theta \in \arg z\}.$$

which is a multi-value function. Recall the notion of the principle argument $\operatorname{Arg} z$. This reduces the multi-value function into a single value function.

Definition 3.2 (Logarithm function)**.** Define the single-valued logarithm function $\log : C - \{0\} \to \mathbb{C}$ by

$$\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z.$$

We also call this the principle value of $\log z$. □

Theorem 3.2. *The function* $\operatorname{Log} z$ *is analytic on the* cut complex plane $\mathbb{C} \setminus (-\infty, 0]$ *and furthermore*

$$\frac{\mathrm{d}}{\mathrm{d}z} \operatorname{Log} z = \frac{1}{z}.$$

Proof. Let $z_0 \in \mathbb{C} \setminus (-\infty, 0]$. Let $w = \operatorname{Log} z$ and $z_0 = \operatorname{Log} z_0$, such that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}z} \operatorname{Log} z \bigg|_{z=z_0} &= \lim_{z \to z_0} \frac{w - w_0}{z - z_0} \\ &= \lim_{z \to z_0} \frac{w - w_0}{e^w - e^{w_0}} \end{aligned}$$

Note that

$$\lim_{z \to z_0} \frac{e^w - e^{w_0}}{w - w_0} = \frac{\mathrm{d}}{\mathrm{d}w} e^w \bigg|_{w=w_0} = e^{w_0} = z_0.$$

∎

Definition 3.3 (Branches)**.** $F(z)$ is said to be a *branch* of a multiple-valued function $f(z)$ in a domain D if

(i) $F(z)$ is single-valued and analytic on D and

(ii) for all $z \in D$, $F(z)$ is one of the values of $f(z)$. □

This means that $\operatorname{Log} z$ is a branch of $\log z$ in the cut complex plane, called the *principle branch* of $\log z$. We can define other branches of $\log z$. Define

$$L_\alpha(z) = \ln|z| + i\theta$$

where $\theta \in \arg z \cap (\alpha, \alpha + 2\pi)$. The ray $\theta = \alpha$ is called the *branch cut* for L_α. Each L_α is analytic on the complex plane without the ray $\theta = \alpha$ and the point 0:

$$\mathbb{C}_\alpha = \mathbb{C} \setminus \{z \mid \operatorname{Arg} z = \alpha\} \setminus \{0\}$$

3.3 Complex exponents

Definition 3.4 (Complex exponents)**.** For $z, c \in \mathbb{C}$ with $z \neq 0$, define

$$z^c = \exp(c \log z).$$

The principal branch of the exponent is defined by

$$\operatorname{Pr}(z^c) = \exp(c \operatorname{Log} z)$$

Theorem 3.3. *The function* $\operatorname{Pr}(z^c)$ *is analytic on the cut complex plane* $\mathbb{C} - (-\infty, 0]$ *and*

$$\frac{\mathrm{d}}{\mathrm{d}z} \operatorname{Pr}(z^c) = c \operatorname{Pr}(z^{c-1}).$$

Proof. We have via the chain rule

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}z} \operatorname{Pr}(z^c) &= \exp(c \operatorname{Log} z) \frac{c}{z} \\
&= \exp(c \operatorname{Log} z) \frac{c}{\exp(\operatorname{Log} z)} \\
&= c \exp(c - 1 \operatorname{Log} z) \\
&= c \operatorname{Pr}(z^{c-1}).
\end{aligned}$$

■

More generally for each $\alpha \in \mathbb{R}$, the function defined on $\mathbb{C}_\alpha$

$$F_{\alpha,c}(z) = \exp(c L_\alpha(z))$$

is a branch of z^c and

$$\frac{\mathrm{d}}{\mathrm{d}z} F_{\alpha,c}(z) = c F_{\alpha,c-1}(z).$$

3.4 Trigonometric functions

The following follows directly from our definition of the complex exponential function.

Definition 3.5 (Sine and cosine)**.** For $z \in \mathbb{C}$, define

$$\begin{aligned}
\sin z &= \frac{1}{2i}\left(e^{iz} - e^{-iz}\right) \\
\cos z &= \frac{1}{2}\left(e^{iz} + e^{-iz}\right).
\end{aligned}$$

□

The usual trigonometric identities also hold in the complex plane, as well as the familiar derivatives.

Theorem 3.4. *We have* $\frac{\mathrm{d} \cos z}{\mathrm{d}z} = -\sin z$ *and* $\frac{\mathrm{d} \sin z}{\mathrm{d}z} = \cos z$*.*

The other trigonometric functions like $\tan z$, $\sec z$, are all defined as per usual from $\sin z$ and $\cos z$.

3.5 Hyperbolic functions

Definition 3.6. For $z \in C$, define

$$\sinh z = \frac{1}{2}(e^z - e^{-z})$$
$$\cosh z = \frac{1}{2}(e^z + e^{-z}). \qquad \square$$

Theorem 3.5. *We have* $\frac{\mathrm{d} \sinh z}{\mathrm{d}z} = \cosh z$ *and* $\frac{\mathrm{d} \cosh z}{\mathrm{d}z} = \sinh z$.

4 Integrals

4.1 Integration

Definition 4.1. Let $w(t) = u(t) + iv(t)$ be a complex valued function of a real variable. Define the integral of w to be

$$\int_a^b w(t)\,\mathrm{d}t = \int_a^b u(t)\,\mathrm{d}t + i\int_a^b v(t)\,\mathrm{d}t. \qquad \square$$

Theorem 4.1. *Suppose* $F'(t) = f(t)$*. Then*

$$\int_a^b f(t)\,\mathrm{d}t = F(b) - F(a).$$

Theorem 4.2. *If* $w : [a, b] \to \mathbb{C}$*, then*

$$\left|\int_a^b w(t)\,\mathrm{d}t\right| \leq \int_a^b |w(t)|\,\mathrm{d}t.$$

Proof. Let $re^{i\theta} = \int_a^b w(t)\,\mathrm{d}t$. Now

$$\begin{aligned} r &= \left|\int_a^b w(t)\,\mathrm{d}t\right| \\ &= e^{-i\theta}\int_a^b w(t)\,\mathrm{d}t \\ &= \int_a^b \Re[e^{-i\theta}w(t)]\,\mathrm{d}t \\ &\leq \left|\int_a^b \Re[e^{-i\theta}w(t)]\,\mathrm{d}t\right| \\ &\leq \int_a^b \left|\Re[e^{-i\theta}w(t)]\,\mathrm{d}t\right| \\ &\leq \int_a^b \left|e^{-i\theta}w(t)\right|\mathrm{d}t \\ &= \int_a^b |w(t)|\,\mathrm{d}t. \end{aligned}$$

$\blacksquare$

Definition 4.2 (Simple curves). For a curve $\gamma : [a, b] \to \mathbb{C}$, we call it *simple* if for $t_1, t_2 \in (a, b)$, $t_1 \neq t_2 \implies \gamma(t_1) \neq \gamma(t_2)$. In other words it does not cross itself except possibly at the endpoints. □

Definition 4.3 (Closed curves). For a curve $\gamma : [a, b] \to \mathbb{C}$, we call it *closed* if $\gamma(a) = \gamma(b)$. □

Definition 4.4 (Smooth curves). For a curve $\gamma : [a, b] \to \mathbb{C}$, we call it *smooth* if $\gamma'(t)$ exists and is continuous on $[a, b]$, and $\gamma'(t) \neq 0$ for all $t \in (a, b)$. □

Definition 4.5 (Length of a smooth curve). The *length* of a smooth curve $\gamma : [a, b] \to \mathbb{C}$ is defined by

$$\int_a^b |\gamma'(t)| \, \mathrm{d}t \, .$$

□

Definition 4.6 (Path integrals). Let S be an open set and let $\gamma : [a, b] \to \mathbb{C}$ be a smooth curve in $\gamma : [a, b] \to \mathbb{C}$ be a smooth curve in S. If $f : S \to \mathbb{C}$ is continuous, then the *integral of f along γ* is defined as

$$\int_\gamma f(z) \, \mathrm{d}z = \int_a^b f(\gamma(t))\gamma'(t) \, \mathrm{d}t \, .$$

□

Theorem 4.3. *Let $\gamma : [a, b] \to \mathbb{C}$ be a smooth curve, and let $\phi[c, d] \to [a, b]$ be such that*

(i) $\phi'(t)$ exists and is continuous on $[c, d]$, and

(ii) $\phi(c) = a$ and $\phi(d) = b$.

Let $\alpha(t) = \gamma(\phi(t))$. In other words, α is a re-parametrisation of γ. Then for any continuous function f,

$$\int_\gamma f(z) \, \mathrm{d}z = \int_\alpha f(z) \, \mathrm{d}z \, .$$

Proof. The last step uses a change of variables $s = \gamma(t)$:

$$\begin{aligned}
\int_\alpha f(z) \, \mathrm{d}z &= \int_c^d f[\alpha(t)]\alpha'(t) \, \mathrm{d}t \\
&= \int_c^d f[\gamma(\phi(t))]\gamma'(\phi(t))\phi'(t) \, \mathrm{d}t \\
&= \int_a^b f[\gamma(s)]\gamma'(s) \, \mathrm{d}s \\
&= \int_\gamma f(z) \, \mathrm{d}z \, .
\end{aligned}$$

■

Definition 4.7 (Opposite curve). Let $\gamma : [a, b] \to \mathbb{C}$ be a curve. Define its *opposite curve* as

$$(-\gamma)(t) = \gamma(-t).$$

□

Theorem 4.4. *For any smooth curve γ, and function f,*

$$\int_{-\gamma} f(z)\,\mathrm{d}z = -\int_{\gamma} f(z)\,\mathrm{d}z\,.$$

Proof. Perform a change of variable in the integral of $s = -t$. ■

Definition 4.8 (Contours)**.** A *contour* Γ is a sequence of smooth curves $\{\gamma_1, \ldots, \gamma_n\}$ such that the end point of γ_k coincides with the start point of γ_{k+1}. We write $\Gamma = \gamma_1 + \cdots + \gamma_n$. □

Integrals along contours are defined as the piecewise sum of integrals over the constituent curves. The same goes for other notions like length, opposite contours, etc.

Theorem 4.5 (ML inequality)**.** *Suppose that f is continuous on an open set containing a contour Γ and $|f(z)| \leq M$ for all $z \in \operatorname{ran}\Gamma$. Let L be the length of γ. Then*

$$\left|\int_{\gamma} f(z)\,\mathrm{d}z\right| \leq ML.$$

Proof. First assume that $\gamma : [a, b] \to \mathbb{C}$ is a smooth curve. Then

$$\begin{aligned}
\left|\int_{\gamma} f(z)\,\mathrm{d}z\right| &= \left|\int_a^b f[\gamma(t)]\gamma'(t)\,\mathrm{d}t\right| \\
&\leq \int_a^b |f[\gamma(t)\gamma'(t)|\,\mathrm{d}t \\
&\leq M\int_a^b |\gamma'(t)|\,\mathrm{d}t \\
&= ML.
\end{aligned}$$

For the case when Γ is a contour where $\gamma = \gamma_1 + \cdots + \gamma_n$, then

$$\begin{aligned}
\left|\int_{\Gamma} f(z)\,\mathrm{d}z\right| &= \left|\sum_k \int_{\gamma_k} f(z)\,\mathrm{d}z\right| \\
&\leq \sum_k \left|\int_{\gamma_k} f(z)\,\mathrm{d}z\right| \\
&\leq \sum_k ML(\gamma_k) \\
&= ML.
\end{aligned}$$

■

Example 4.1. Let $\gamma(t) = 2e^{it}$ and $f(z) = \frac{e^z}{z^2+1}$. Apply the ML-inequality on the integral $\int_\gamma f(z)\,\mathrm{d}z$.

First of all, for all $z \in \operatorname{ran}\gamma$, $|z| = 2$. Thus $|e^z| \leq e^2$. Also, $\left|z^2 + 1\right| = \left|z^2 - (-1)\right| \geq \left|\left|z^2\right| - |-1|\right| = 3$.

Putting it all together,

$$\left|\int_{\gamma} f(z)\,\mathrm{d}z\right| \leq \frac{e^2}{3} \cdot 4\pi.$$

◊

4.2 Antiderivatives

Definition 4.9 (Antiderivatives)**.** Let f be a continuous function on an open domain D. A function F such that $F'(z) = f(z)$ for all $z \in D$ is called an *antiderivative* of f in D. □

Note that if f has an antiderivative F, then since F is analytic, so must f. This also means that the domain that f is defined on must be an open set to begin with.

Theorem 4.1. *Suppose f has an antiderivative F on an open domain D. If Γ is a contour in D with endpoints z_1 and z_2, then*

$$\int_\Gamma f(z)\,\mathrm{d}z = F(z_2) - F(z_1).$$

In particular, if Γ is a closed contour ($z_1 = z_2$), then

$$\int_\Gamma f(z)\,\mathrm{d}z = 0.$$

Proof. Let $\Gamma = \gamma_1 + \cdots + \gamma_n$ where $\gamma_j : [a_{j-1}, a_j] \to \mathbb{C}$ is a smooth curve. For each $i \le j \le n$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} F[\gamma_j(t)] = F'[\gamma_j(t)]\gamma_j'(t) = f[\gamma_j(t)]\gamma_j'(t).$$

Thus,

$$\begin{aligned}\int_{\gamma_j} f(z)\,\mathrm{d}z &= \int_{a_{j-1}}^{a_j} f[\gamma_j(t)]\gamma_j'(t)\,\mathrm{d}t \\ &= F[\gamma_j(a_j)] - F[\gamma_j(a_{j-1})].\end{aligned}$$

Then,

$$\begin{aligned}\int_\Gamma f(z)\,\mathrm{d}z &= \int_{\gamma_1} f(z)\,\mathrm{d}z + \cdots + \int_{\gamma_n} f(z)\,\mathrm{d}z \\ &= F(z_2) - F(z_1).\end{aligned}$$

■

Theorem 4.2. *Let f be continuous on an open domain D. The following statements are equivalent:*

(i) f has an antiderivative in D,

(ii) for any closed contour Γ in D, $\int_\Gamma f(z)\,\mathrm{d}z = 0$,

(iii) the contour integrals of f in D are path-independent.

Proof. From theorem 4.1 we have shown *(i)* $\implies$ *(ii)* and *(i)* $\implies$ *(iii)*.

Now we show *(ii)* $\implies$ *(iii)*. Let Γ_1 and Γ_2 in D be contours with the same endpoints. Then $\Gamma_1 + (-\Gamma_2)$ is a closed contour in D. By *(ii)*,

$$\int_{\Gamma_1} f(z)\,\mathrm{d}z - \int_{\Gamma_2} f(z)\,\mathrm{d}z = 0.$$

Finally we show *(iii)* $\implies$ *(i)*. Take $z_0 \in D$. For any $z_1 \in D$, define

$$F(z_1) = \int_\Gamma f(z)\,\mathrm{d}z$$

where Γ is a contour in D joining z_0 to z_1. This is well defined by *(iii)*.

Since f is continuous at z_1, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$|z - z_1| < \delta \implies |f(z) - f(z_1)| < \epsilon.$$

Since f is analytic, there is some $h \neq 0$ and $|h| < \delta$ such that the line segment $[z_1, z_1 + h] \subseteq D$. Then,

$$\begin{aligned}
\frac{F(z_1+h) - F(z_1)}{h} &= \frac{1}{h}\left(\int_{\gamma+[z_1,z_1+h]} f(z)\,\mathrm{d}z - \int_\gamma f(z)\,\mathrm{d}z\right) \\
&= \frac{1}{h}\int_{[z_1,z_1+h]} f(z)\,\mathrm{d}z \\
\frac{F(z_1+h) - F(z_1)}{h} - f(z_1) &= \frac{1}{h}\int_{[z_1,z_1+h]} f(z)\,\mathrm{d}z - f(z_1)\frac{1}{h}\int_{[z_1,z_1+h]} 1\,\mathrm{d}z \\
&= \frac{1}{h}\int_{[z_1,z_1+h]} f(z) - f(z_1)\,\mathrm{d}z\,.
\end{aligned}$$

The limit of the last term as $h \to 0$ is actually 0, because by the ML-inequality,

$$\left|\frac{1}{h}\int_{[z_1,z_1+h]} f(z) - f(z_1)\,\mathrm{d}z\right| \leq \frac{1}{|h|}\epsilon|h| = \epsilon.$$

This means $F'(z_1) = f(z_1)$. ■

Theorem 4.3 (Cauchy-Goursat theorem for rectangles). *Let f be a function which is analytic on (including the interior) a rectangle R, with a positively oriented boundary ∂R. Then*

$$\int_{\partial R} f(z)\,\mathrm{d}z = 0.$$

Proof. Divide R into 4 congruent rectangles, R^1 to R^4. One of the rectangles among them has the greatest integral, call it R_1, such that

$$\left|\int_{\partial R_1} f(z)\,\mathrm{d}z\right| = \max_{1\leq k\leq 4}\left|\int_{\partial R^k} f(z)\,\mathrm{d}z\right|.$$

This gives

$$\begin{aligned}
\left|\int_{\partial R} f(z)\,\mathrm{d}z\right| &\leq \sum_{k=1}^{4}\left|\int_{\partial R^k} f(z)\,\mathrm{d}z\right| \\
&\leq 4\left|\int_{\partial R_1} f(z)\,\mathrm{d}z\right|
\end{aligned}$$

Do the same step for R_1 to obtain a smaller rectangle R_2. Continuing this way, we obtain a sequence of rectangles

$$R \subset R_1 \subset R_2 \subset \cdots$$

such that

$$\left| \int_{\partial R_{k-1}} f(z)\,\mathrm{d}z \right| \leq 4 \left| \int_{\partial R_k} f(z)\,\mathrm{d}z \right|.$$

This means

$$\left| \int_{\partial R} f(z)\,\mathrm{d}z \right| \leq 4^n \left| \int_{\partial R_n} f(z)\,\mathrm{d}z \right|$$

We claim that there is some point z_0 that is common to all rectangles R_n. We briefly sketch a proof. Form an sequence of closed intervals $[a_n, b_n]$ as follows. The interval $[a_i, b_i]$ is either the left or right half of the previous interval $[a_{i-1}, b_{i-1}]$. Then note that the sequence (a_n) and (b_n) are both bounded monotonic sequences. Thus they have a limit. The length of the interval goes to 0, and so they must tend to the same limit. Using this result, apply it to the two edges that form the rectangles R_n.

Next, let d_n denote the length of the diagonal of R_n and l_n denote the length of ∂R_n. Now let $\epsilon > 0$. As the size of the rectangles are decreasing, and yet they also contain z_0, there exists some rectangle $R_m \subseteq B(z_0, \delta)$. Consequently, for all $z \in \partial R_m$,

$$\begin{aligned} 0 < |z - z_0| < \delta &\implies \left| f'(z) - \frac{f(z) - f(z_0)}{z - z_0} \right| < \epsilon \\ &\implies |f(z) - f(z_0) - (z - z_0) f'(z_0)| < \epsilon d_m \end{aligned}$$

By the ML-inequality,

$$\begin{aligned} \left| \int_{\partial R_m} f(z) - f(z_0) - (z - z_0) f'(z_0)\,\mathrm{d}z \right| &\leq \epsilon d_m l_m \\ &= \epsilon \frac{d_0}{2^m} \frac{l_0}{2^m}. \end{aligned}$$

Note that $f(z_0)$ and $f'(z_0)$ are constants in the integral, and $(z - z_0)$ has an antiderivative. Thus in fact this reduces to

$$\left| \int_{\partial R_m} f(z)\,\mathrm{d}z \right| \leq \epsilon \frac{d_0 l_0}{4^m}$$

and thus from a result above,

$$\left| \int_{\partial R} f(z)\,\mathrm{d}z \right| \leq \epsilon d_0 l_0$$

Thus as $\epsilon \to 0$ we have

$$\int_{\partial R} f(z)\,\mathrm{d}z = 0.$$

■

Theorem 4.4 (Cauchy-Goursat theorem)**.** *If a function f is analytic at all points on and interior to a simple closed contour Γ, then*

$$\int_{\Gamma} f(z)\,\mathrm{d}z = 0.$$

Proof. Let the region enclosed by Γ be called R. The only difference now is that we have to consider rectangles that have points that are not in R. Call these rectangles that are intersections with an rectangle and R, partial rectangles. We only need to change the upper bound on the integral to take into account the perimeter of these partial rectangles. ■

If a contour is able to be continuously deformed into another contour, always passing through points in which the function is analytic, then the integral does not change.

Theorem 4.5 (Cauchy-Goursat theorem for simply connected domains)**.** *If a function f is analytic in a simply connected domain D, then*

$$\int_\Gamma f(z)\,\mathrm{d}z = 0$$

for every closed contour Γ *in D.*

Proof. If the curve intersects itself a finite number of times, the Cauchy-Goursat theorem can be applied to each of the simple closed contours that it is made up of.

For the infinite case, TODO ■

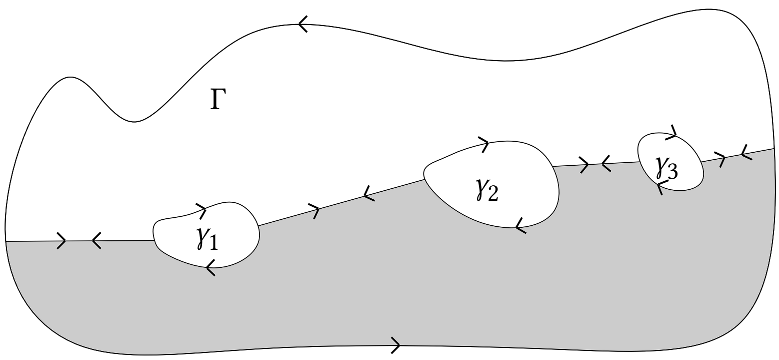
Theorem 4.6 (Cauchy-Goursat theorem for multiply connected domains)**.** *Let*

- Γ *is a simple positively oriented closed contour,*
- $\gamma_1, \dots, \gamma_k$ *are mutually disjoint positively oriented simple closed contours interior to* Γ*,*
- *D refer to the domain consisting of the points inside* Γ *and outside* $\gamma_1, \dots, \gamma_k$*.*

If a function f is analytic on all of these contours as well as in D, then

$$\int_\Gamma f(z)\,\mathrm{d}z + \sum_{n=1}^{k} \int_{\gamma_n} f(z)\,\mathrm{d}z = 0.$$

Proof. Refer to the figure for an example.



Create a new integration path with line segments joining Γ to γ_1, γ_1 to γ_2, so on and so forth, and finally $\gamma_k \longrightarrow \Gamma$ again. This essentially divides the boundary of D into two simple closed contours in which f is analytic in. Apply the Cauchy-Goursat theorem on these two pieces and sum them up. We will find that the integrals along the line segments cancel, leaving us with the contour integrals. ■

Corollary 4.5.1 (Principle of deformation of paths). *Let Γ_1 and Γ_2 be positively oriented simple closed contours with Γ_2 interior to Γ_1. If f is analytic in the closed region consisting of these contours and the region between them, then*

$$\int_{\Gamma_1} f(z)\,\mathrm{d}z = \int_{\Gamma_2} f(z)\,\mathrm{d}z\,.$$

Proof. It follows directly from the previous theorem. ■

Example 4.2. Suppose that Γ is a positively oriented simple closed contour that contains z_0. We want to evaluate

$$\int_{\Gamma} \frac{1}{z - z_0}\,\mathrm{d}z\,.$$

There is a circle with radius r small enough that is interior to Γ. $1/(z - z_0)$ is analytic on the region between the circle and Γ, as well as on these two contours. Therefore

$$\int_{\Gamma} \frac{1}{z - z_0}\,\mathrm{d}z = 2\pi i.$$

◊

Definition 4.10 (Simply connected domains). A domain D is *simply connected* if every simple closed contour in D encloses only points in D. In other words, D has no "holes". □

Example 4.3. The following are simply connected domains:

- Open balls.
- Interiors of simply closed contours.
- The cut complex plane.
- The entire complex plane.

The following are not simply connected domains:

- The annular domain $\{z \mid 1 < |z| < 2\}$.
- The puncture plane $\mathbb{C} \setminus \{0\}$. ◊

Theorem 4.7 (Cauchy-Goursat theorem for simply connected domains). *If f is analytic in a simply connected domain D, then*

$$\int_{\Gamma} f(z)\,\mathrm{d}z = 0$$

Proof. todo ■

Corollary 4.5.2. *If f is analytic in a simply connected domain D, then it has an antiderivative in D.*

4.3 Cauchy's formula

Theorem 4.8 (Cauchy integral formula). *Let* Γ *be a positively oriented simple closed contour and let* f *be analytic within and on* Γ. *Then for any* z_0 *interior to* Γ,

$$f(z_0) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z - z_0}\, \mathrm{d}z$$

Proof. Let $\epsilon > 0$. Since f is continuous at z_0, there is a $\delta > 0$ such that

$$|z - z_0| < \delta \implies |f(z) - f(z_0)| < \frac{\epsilon}{2\pi}.$$

Now choose $0 < r < \delta$ such that the circle $\gamma(t) = z_0 + re^{it}$ is completely interior to Γ. Then the integral evaluated on Γ is equal to the integral evaluated on γ.

For all $z \in \{\gamma\}$, since $r < \delta$, we also have $|z - z_0| < \delta$. This means that

$$\left|\frac{f(z) - f(z_0)}{z - z_0}\right| < \frac{\epsilon}{2\pi r}.$$

Therefore, by the ML inequality,

$$\left|\int_\gamma \frac{f(z) - f(z_0)}{z - z_0}\, \mathrm{d}z\right| \leq \epsilon.$$

We also have

$$\underbrace{\left|\int_\gamma \frac{f(z) - f(z_0)}{z - z_0}\, \mathrm{d}z\right|}_{\leq \epsilon} = \left|\int_\Gamma \frac{f(z)}{z - z_0}\, \mathrm{d}z - f(z_0)2\pi i\right|$$

As $\epsilon \to 0$, we obtain the desired result. ■

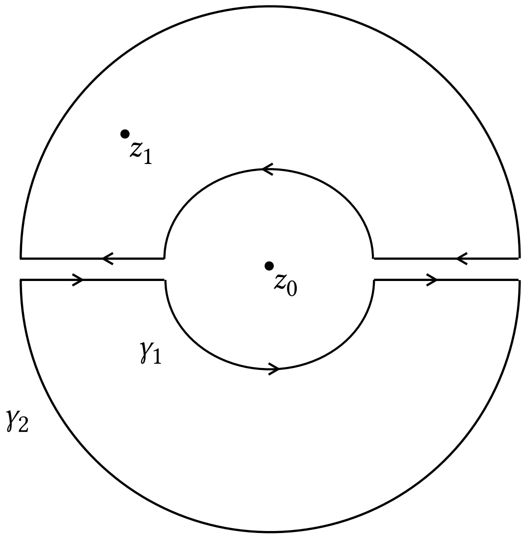
Example 4.4. We want to evaluate $\int_\Gamma \frac{z}{(9-z^2)(z+i)}$ where Γ is the circle centred around the origin with radius 2.

Let $f(z) = \frac{z}{9-z^2}$. Then f is analytic within and on Γ. By Cauchy's integral formula,

$$\int_\Gamma \frac{z}{(9 - z^2)(z + i)}\, \mathrm{d}z = 2\pi i f(-i) = \frac{\pi}{5}.$$

◊

Example 4.5 (Cauchy's formula in an annulus). Let f be analytic in the closed annulus $A = \{z \mid R_1 \leq |z - z_0| \leq R_2\}$ and let z_1 be an interior point of A. Let γ_1 and γ_2 be the positively oriented circles $|z - z_0| = R_1$ and $|z - z_0| = R_2$ respectively.



Refer to the diagram. We split the annulus into two such that the function is analytic within these two new contours. The integral then evaluates to

$$f(z_1) = \frac{1}{2\pi i}\left[\int_{\gamma_2} \frac{f(z)}{z - z_1}\,\mathrm{d}z - \int_{\gamma_1} \frac{f(z)}{z - z_1}\,\mathrm{d}z\right].$$

$\Diamond$

Theorem 4.9 (Cauchy's integral formula for derivatives). *Let Γ be a positively oriented simple closed contour and let f be analytic within and on Γ. Then for any point z_0 interior to Γ,*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_\Gamma \frac{f(z)}{(z - z_0)^{n+1}}\,\mathrm{d}z\,.$$

Proof. We prove this by induction. This holds for $n = 0$ as we showed previously.

Suppose this holds for n. For the case of $n + 1$, first we evaluate

$$\begin{aligned}
f^{(n)}(z_0 + h) - f^{(n)}(z_0) &= \frac{n!}{2\pi i}\int_\Gamma \frac{f(z)}{(z - z_0 - h)^{n+1}} - \frac{f(z)}{(z - z_0)^{n+1}}\,\mathrm{d}z \\
&= \frac{n!}{2\pi i}\int_\Gamma f(z)\frac{w^{n+1} - (w-h)^{n+1}}{w^{n+1}(w-h)^{n+1}}\,\mathrm{d}z \\
&= \frac{n!}{2\pi i}\int_\Gamma f(z)\frac{h\big[(w-h)^n + w(w-h)^{n-1} + \cdots + w^n\big]}{w^{n+1}(w-h)^{n+1}}\,\mathrm{d}z\,.
\end{aligned}$$

where $w = z - z_0$. Next,

$$\begin{aligned}
&\frac{f^{(n)}(z_0 + h) - f^{(n)}(z_0)}{h} - \frac{(n+1)!}{2\pi i}\int_\Gamma \frac{f(z)}{w^{n+2}}\,\mathrm{d}z \\
&\quad = \frac{n!}{2\pi i}\int_\Gamma f(z)\frac{w(w-h)^n + w^2(w-h)^{n-1} + \cdots + w^{n+1} - (n+1)(w-h)^{n+1}}{w^{n+2}(w-h)^{n+1}}\,\mathrm{d}z\,.
\end{aligned}$$

Let us try to simplify the numerator:

$$\begin{aligned}
&w(w-h)^n + \cdots + w^n(w-h) + w^{n+1} - (w-h)^{n+1} - n(w-h)^{n+1} \\
&\quad = w(w-h)^n + \cdots + w^n(w-h) + h(t^n + t^{n-1}(t-h) \cdots + (t-h)^n) - (w-h)^n(w-h) - (n-1)(w-h)^{n+1} \\
&\quad = w(w-h)^n + \cdots + (w-h)h(w^{n-1} + \cdots + (w-h)^{n-1}) + h(t^n + \cdots + (t-h)^n) - (w-h)^{n-1}(w-h) - (n-2)(w-h)^{n+1} \\
&\quad \vdots \\
&= h\{(w-h)^n + [w + (w-h)](w-h)^{n-1} + \cdots + [w^n + w^{n-1}(w-h) + \cdots + (w-h)^n]\}
\end{aligned}$$

Now let $M = \max_{z\in\{\Gamma\}} |f(z)|$, let d be the shortest distance from z_0 to Γ, and let D be the greatest distance from z_0 to Γ. Then for h such that $|h| < d/2$,

$$d < |w| = |z - z_0| < 2D \qquad \frac{d}{2} < |w - h| < 2D$$

By the ML inequality,

$$\begin{aligned}
&= \left| \frac{f^{(n)}(z_0+h) - f^{(n)}(z_0)}{h} - \frac{(n+1)!}{2\pi i} \int_\Gamma \frac{f(z)}{w^{n+2}}\,\mathrm{d}z \right| \\
&= \frac{n!h}{2\pi i} \int_\Gamma f(z) \frac{(w-h)^n + [w + (w-h)](w-h)^{n-1} + \cdots + [w^n + \cdots + (w-h)^n]}{w^{n+2}(w-h)^{n+1}}\,\mathrm{d}z \\
&\le |h| \frac{n!M}{2\pi} \frac{O(D^n)}{O(d^{2n+3})} L(\Gamma)
\end{aligned}$$

which goes to 0 as $h \to 0$, since all the other terms are constants. ∎

Corollary 4.5.3. *If f is analytic in a domain D, then all its derivatives exist and are analytic in D. In particular, if $f = u + iv$, then u and v have continuous partial derivatives of all orders in D.*

Theorem 4.10 (Morera's theorem)**.** *If f is continuous on a domain D and $\int_\Gamma f(z)\,\mathrm{d}z = 0$ for every closed contour Γ in D, then f is analytic in D.*

Proof. By theorem 4.1, f has an antiderivative F in D. By the previous corollary $f = F'$ is analytic in D. ∎

Theorem 4.11 (Cauchy's inequality)**.** *Let C be a circle centred at z_0 with radius R. Suppose f is a function that is analytic within and on C. Denote $M = \max_{z\in\{C\}} |f(z)|$. Then*

$$\left|f^{(n)}(z_0)\right| \le \frac{n!M}{R^n}.$$

Proof. This is an immediate consequence of Cauchy's formula:

$$\begin{aligned}
\left|f^{(n)}(z_0)\right| &= \left| \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}}\,\mathrm{d}z \right| \\
&\le \frac{n!}{2\pi} \frac{M}{R^{n+1}} 2\pi R \\
&= \frac{n!M}{R^n}.
\end{aligned}$$

∎

Theorem 4.12 (Liouville's theorem)**.** *If an entire function f is bounded, then it must be a constant function.*

Proof. Since f is bounded, there exists M such that $|f(z)| \le M$ for all $z \in \mathbb{C}$. By Cauchy's inequality,

$$|f'(z)| \le \frac{M}{R}.$$

Now this goes to 0 as $R \to \infty$. As this holds for arbitrary z, thus we conclude that $f'(z) = 0$ for all z. ∎

Theorem 4.13 (The Fundamental Theorem of Algebra). *Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial, then $p(z) = 0$ has a solution in $\mathbb{C}$.*

Proof. Suppose not. Suppose that instead $p(z) \neq 0$ for all $z \in \mathbb{C}$. Then $1/p(z)$ would be an entire function.

Next we show that $1/p(z)$ is bounded. Let $M = \max(1, \|a_0\|, \ldots, |a_{n-1}|)$ and $R = 2nM > 1$. Then for all $|z| > R$, $1 \leq j \leq n$, we have

$$\left|\frac{a_{n-j}}{z^j}\right| \leq \frac{M}{|z|} < \frac{M}{2nM} = \frac{1}{2n},$$

such that

$$\begin{aligned}\left|\frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n}\right| &\leq \left|\frac{a_{n-1}}{z}\right| + \cdots + \left|\frac{a_0}{z^n}\right| \\ &\leq \frac{1}{2n} + \cdots + \frac{1}{2n} \\ &= \frac{1}{2}.\end{aligned}$$

Now this means that

$$\begin{aligned}|p(z)| &= |z^n|\left|1 + \left(\frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n}\right)\right| \\ &\geq |z^n|\left|1 - \left|\frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n}\right|\right| \\ &\geq |z^n|\left(1 - \frac{1}{2}\right) \\ &> \frac{R^n}{2}\end{aligned}$$

Thus $1/p(z)$ is bounded by $\frac{2}{R^n}$ for the case where $|z| > R$. However the closed ball $B(0, R)$ is compact, so again $1/p(z)$ has to be bounded there as well. Hence $1/p(z)$ is bounded on the entire complex plane. By Liouville's theorem this would suggest that $p(z)$ is a constant function which is a contradiction. ■

5 Sequences and series

The ideas are very similar to those real analysis. Many of the theorems will be stated without proof, refer to the real analysis notes for proofs. Many times it is simply applying the real analytic methods onto the real and complex components individually then putting them back.

5.1 Sequences

Definition 5.1 (Sequences). A sequence can be formally defined by a function $\mathbb{N} \to \mathbb{C}$. We shall denote a sequence of complex numbers $z_1, z_2, \ldots$ by $(z_n)_{n=1}^{\infty}$ or as short by (z_n). □

Definition 5.2 (Limits). We say that the sequence (z_n) has a *limit* at z if

$$\forall \epsilon > 0, \exists N \in \mathbb{N}\ [n \geq N \implies |z_n - z| < \epsilon].$$

We write $\lim_{n\to\infty} z_n = z$. We also say it that (z_n) *converges* to z. If a sequence does not have a limit then we say it *diverges*. □

Theorem 5.1. *If a sequence is convergent then its limit is unique.*

Theorem 5.2. *If a sequence is convergent then it is bounded.*

Theorem 5.3. *If $z \in \mathbb{C}$ and $|z| < 1$, then $\lim_{n\to\infty} z^n = 0$.*

Proof. Let $\epsilon > 0$. Let $r = |z|$. Then we know from real analysis that $\lim_{n\to\infty} r^n = 0$, so there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|r^n| < 0$. Then it follows that $|z^n| < \epsilon$ as well. ■

Theorem 5.4. *For a sequence (z_n), if $z_n = x_n + iy_n$, then*

$$\lim_{n\to\infty} z_n = x + iy \iff \lim_{n\to\infty} x_n = x \wedge \lim_{n\to\infty} x_n = y.$$

Proof. See theorem 2.1. ■

Theorem 5.5. *Let (z_n) and (w_n) be sequences, and $\lim_{n\to\infty} z_n = z$ and $\lim_{n\to\infty} w_n = w$. Then*

(i) $\lim_{n\to\infty}(z_n + w_n) = z + w$.

(ii) $\lim_{n\to\infty}(z_n w_n) = zw$.

(iii) $\lim_{n\to\infty} \frac{z_n}{w_n} = \frac{z}{w}$ *if $w_n \neq 0$ for all n.*

Definition 5.3 (Cachy sequences). A sequence (z_n) is called *Cauchy* if

$$\forall \epsilon > 0, \exists N \in \mathbb{N},\ [n, m \geq N \implies |z_n - z_m| < \epsilon]$$

□

Theorem 5.6 (Cauchy criterion). *A sequence (z_n) is convergent iff it is Cauchy.*

5.2 Series

Definition 5.4 (Series). Given a sequence (z_n), form the sequence of partial sums where $S_n = z_1 + \cdots + z_n = \sum_{i=1}^{n} z_i$. We call S a *series* and write $\sum_{n=1}^{\infty} z_n$. □

Since a series is also a sequence, the same theorems and definitions for convergence/divergence apply to it.

Theorem 5.7. *If $\sum_{z=1}^{\infty} {}_n$ converges, then $\lim_{n\to\infty} z_n = 0$.*

Definition 5.5 (Absolute convergence). If $\sum_{n=1}^{\infty} |z_n|$ converges, then we say that $\sum_{n=1}^{\infty} z_n$ converges *absolutely*. □

Theorem 5.8. *If a series converges absolutely then it converges.*

Theorem 5.9 (Comparison test). *If* $|z_n| \le a_n$*, and* $\sum_{n=1}^{\infty} a_n$ *converges, then* $\sum_{n=1}^{\infty} z_n$ *converges absolutely.*

Theorem 5.10 (Geometric series). *If* $z \in B(0,1)$*, then*

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}.$$

Proof. Each partial sum is given by

$$S_n = 1 + z + z^2 + \cdots + z^{n-1} = \frac{1-z^n}{1-z}.$$

Since $|z| < 1$, $\lim_{n\to\infty} z^n = 0$, so

$$\lim_{n\to\infty} S_n = \frac{1}{1-z}.$$

■

Definition 5.6 (Power series). A series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

if called a *power series.* □

Theorem 5.11. *If* $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ *converges at* $z = z_1$*, then it converges absolutely for all* z *such that* $|z - z_0| < |z_1 - z_0|$.

Theorem 5.12 (Convergence radius). *For any power series* $\sum_{n=0}^{\infty} a_n (z-z_0)^n$*, there is an unique* $0 \le R \le \infty$ *such that*

(i) the series converges absolutely for all $|z - z_0| < R$*,*

(ii) the series diverges for all z *such that* $|z - z_0| > R$*,*

(iii) and no conclusion otherwise.

Theorem 5.13 (Ratio test). *If* $\lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| = L$*, then*

(i) if $L < 1$*, then* $\sum_{n=1}^{\infty} z_n$ *converges absolutely,*

(ii) if $L > 1$*, then* $\sum_{n=1}^{\infty} z_n$ *diverges,*

(iii) otherwise no conclusion can be made.

Furthermore, the convergence radius $R = 1/L$.

Theorem 5.14 (Cauchy-Hadamard). *For any power series* $\sum_{n=0}^{\infty} a_n (z-z_0)^n$*, its convergence radius is given by*

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}.$$

5.3 Sequences of functions

Definition 5.7 (Pointwise convergence)**.** Let (f_n) be a sequence of functions defined on a subset $D \subseteq \mathbb{C}$. Suppose that for all $z \in D$, the sequence $(f(z))$ converges. Then we define a function $f : D \to \mathbb{C}$ by

$$f(z) = \lim_{n\to\infty} f_n(z)$$

and say that (f_n) converges to f *pointwise* in D. □

Definition 5.8 (Uniform convergence)**.** We say that a sequence of functions (f_n) converge to f *uniformly* if

$$\forall \epsilon > 0, \exists N \in \mathbb{N},\ [n \geq N \implies |f_n(z) - f(z)| < \epsilon].$$

□

The difference between the two is that for uniform convergence, the same value of N works for all points $z \in D$.

Example 5.1. Let $f_n(z) = z^n$. Then

- $f_n \to 0$ pointwise on $B(0, 1)$.
- $f_n \to 0$ uniformly on $\overline{B(0, r)}$ where $0 < r < 1$.

◊

Theorem 5.15. *Let (f_n) be a sequence of functions. If $\lim_{n\to\infty} f_n = f$ uniformly and each f_n is continuous then f is also continuous.*

Theorem 5.16. *Let Γ be a contour and let (f_n) be a sequence of functions continuous on $\{\Gamma\}$. If (f_n) converges uniformly to f on $\{\Gamma\}$, then*

$$\lim_{n\to\infty} \int_\Gamma f_n(z)\,\mathrm{d}z = \int_\Gamma \lim_{n\to\infty} f_n(z)\,\mathrm{d}z = \int_\Gamma f(z)\,\mathrm{d}z$$

Proof. Let $\epsilon > 0$. Then there is $N \in \mathbb{N}$ such that

$$n \geq N \implies |f_n(z) - f(z)| < \frac{\epsilon}{L(\Gamma)}.$$

By the ML-inequality this means

$$\left| \int_\Gamma f_n(z) - f(z)\,\mathrm{d}z \right| < \frac{\epsilon}{L(\Gamma)} L(\Gamma) = \epsilon.$$

■

Theorem 5.17. *Let (f_n) be a sequence of analytic functions on a domain D. If (f_n) converges uniformly to f on D, then f is analytic in D.*

Proof. Take $z_0 \in D$. Since D is open, there is $r > 0$ such that $B(z_0, r) \subseteq D$. Now let Γ be a closed contour in $B(z_0, r)$. Then

$$\lim_{n\to\infty} \int_\Gamma f_n(z)\,\mathrm{d}z = \int_\Gamma f(z)\,\mathrm{d}z\,.$$

Since each f_n is analytic, $\int_\Gamma f_n(z)\,\mathrm{d}z = 0$. Therefore the integral above evaluates to 0. Since each f_n is continuous as well, f is also continuous. Then Morera's theorem says that f is analytic in $B(z_0, r)$. The choice of z_0 is arbitrary, so this means that f is in fact analytic in the whole of D. ■

5.4 Series of functions

Definition 5.9 (Uniform convergence)**.** We say that the series of functions $\sum_{n=1}^{\infty} f_n(z)$ converges to $S(z)$ *uniformly* if the sequence of partial sums $S_n(z) = \sum_{k=1}^{n} f_k(z)$ converges to $S(z)$ uniformly. □

Theorem 5.18 (Interchangibility)**.** *If a series of functions converges uniformly on a contour* Γ*, then we can interchange the summation with the integral.*

$$\sum_{n=1}^{\infty} \int_\Gamma f_n(z)\,\mathrm{d}z = \int_\Gamma \sum_{n=1}^{\infty} f_n(z)\,\mathrm{d}z$$

Theorem 5.19 (Weierstrass M-test)**.** *Let* $\sum_{n=1}^{\infty} M_n$ *be a convergent series of positive numbers. Let* $(f_n(z))$ *be a sequence of functions on a domain* D *where* $|f_n(z)| \le M_n$*. Then* $\sum_{n=1}^{\infty} f_n(z)$ *converges uniformly and absolutely on* D*.*

Lemma 5.1. *Let* R *be the radius of convergence of the geometric series* $\sum_{n=0}^{\infty} a_n(z-z_0)^n$*. For each* $0 < R' < R$*, the series converges uniformly on* $\overline{B(z_0, R')}$*.*

Proof. Take z_1 such that $R' < |z_1 - z_0| < R$. Then $\sum_{n=0}^{\infty} a_n(z_1 - z_0)^n$ converges. This means that it is bounded, so there exists $M > 0$ such that $|a_n(z_1 - z_0)^n| \le M$.

Let $z \in \overline{B(z_0, R')}$. Since $|z - z_0| \le R_1 < |z_1 - z_0|$, so

$$\left|\frac{z - z_0}{z_1 - z_0}\right| \le \frac{R_1}{|z_1 - z_0|} < 1.$$

Now let $r = \frac{R_1}{|z_1 - z_0|}$, we have

$$|a_n(z-z_0)^n| = |a_n(z_1 - z_0)^n| \left|\frac{z - z_0}{z_1 - z_0}\right|^n \le Mr^n$$

Since $|r| < 1$, the series $\sum_{n=0}^{\infty} Mr^n$ converges. By the Weierstrass M-test, the series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges uniformly as well. ■

Theorem 5.20. *Let* R *be the radius of convergence of* $S(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$*. Then*

(i) $S(z)$ *is an analytic function on* $B(z_0, R)$*.*

(ii) If Γ *is a contour in* $B(z_0, R)$ *and* $g(z)$ *is continuous on* $\{\Gamma\}$*, then*

$$\int_\Gamma g(z) \sum_{n=0}^{\infty} a_n(z-z_0)^n\,\mathrm{d}z = \sum_{n=0}^{\infty} \int_\Gamma g(z) a_n(z-z_0)^n\,\mathrm{d}z$$

(iii)

$$\frac{\mathrm{d}}{\mathrm{d}z} \sum_{n=0}^{\infty} a_n(z-z_0)^n = \sum_{n=1}^{\infty} a_n n(z-z_0)^{n-1}$$

Proof. Denote $S_n = \sum_{k=0}^{n} a_k(z-z_0)^k$.

(i) Let $z_1 \in B(z_0, R)$. Choose r such that $|z_1 - z_0| < r < R$. Then by lemma 5.1, $S_n(z)$ converges uniformly on $\overline{B(z_0, r)}$ to $S(z)$. By theorem 5.17, uniform convergence preserves analyticity on a domain, so $S(z)$ is analytic on $B(z_0, r)$. In particular, $S(z)$ is analytic at z_1. Since this is true for all points in $B(z_0, R)$, therefore $S(z)$ is analytic in $B(z_0, R)$.

(ii) Choose $0 < r < R$ such that $\{\Gamma\} \subseteq \overline{B(z_0, r)}$. Note that since $g(z)$ is continuous and $\overline{B(z_0, r)}$ is compact, therefore $g(z)$ must be bounded. Then we can easily show that $g(z)S_n(z)$ converges uniformly to $g(z)S(z)$. Thus,

$$\begin{aligned}\lim_{n\to\infty}\sum_{k=0}^{n}\int_\Gamma g(z)a_n(z-z_0)^n\,\mathrm{d}z &= \lim_{n\to\infty}\int_\Gamma\sum_{k=0}^{n}g(z)a_n(z-z_0)^n\,\mathrm{d}z\\ &= \int_\Gamma\lim_{n\to\infty}\sum_{k=0}^{n}g(z)a_n(z-z_0)^n\,\mathrm{d}z\\ &= \int_\Gamma g(z)S(z)\,\mathrm{d}z\,.\end{aligned}$$

(iii) Let $z_1 \in B(z_0, R)$. Let γ be a positively oriented circle centred at z such that $\{\gamma\} \subseteq B(z_0, R)$. which from Cauchy's integral formula is the expression for $S'(z)$. From Cauchy's formula first note that

$$\frac{\mathrm{d}}{\mathrm{d}z}(z-z_0)^n\bigg|_{z=z_1} = \frac{1}{2\pi i}\int_\gamma\frac{(z-z_0)^n}{(z-z_1)^2}.$$

Again by Cauchy's integral formula,

$$\begin{aligned}S'(z_1) &= \frac{1}{2\pi i}\int_\gamma\frac{S(z)}{(z-z_1)^2}\,\mathrm{d}z\\ &= \sum_{n=0}^{\infty}a_n\frac{1}{2\pi i}\int_\gamma\frac{(z-z_0)^n}{(z-z_1)^2}\,\mathrm{d}z\\ &= \sum_{n=0}^{\infty}a_n\frac{\mathrm{d}}{\mathrm{d}z}(z-z_0)^n\bigg|_{z=z_1}\\ &= \sum_{n=0}^{\infty}a_n n(z_1-z_0)^{n-1}\end{aligned}$$

■

Theorem 5.21 (Taylor's theorem). *If f is analytic in an open ball $B(z_0, R)$, then*

$$f(z) = \sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

which is called the Taylor series *of f at z_0.*

Proof. Let $z \in B(z_0, r)$. Choose r such that $|z - z_0| < r < R$ and let γ be the positively oriented circle $|w - z_0| = r$. For $w \in \{\gamma\}$, we have

$$\begin{aligned}\frac{1}{w-z} &= \frac{1}{(w-z_0)-(z-z_0)}\\ &= \frac{1}{w-z_0}\frac{1}{1-\frac{z-z_0}{w-z_0}}\\ &= \frac{1}{w-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^n.\end{aligned}$$

This means that

$$\frac{f(w)}{w-z} = \sum_{n=0}^{\infty}f(w)\frac{(z-z_0)^n}{(w-z_0)^{n+1}}.$$

By Cauchy's integral formula,

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,\mathrm{d}w \\
&= \frac{1}{2\pi i}\int_\gamma \sum_{n=0}^\infty f(w)\frac{(z-z_0)^n}{(w-z_0)^{n+1}}\,\mathrm{d}w \\
&= \sum_{n=0}^\infty \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-z_0)^{n+1}}\,\mathrm{d}w(z-z_0)^n \\
&= \sum_{n=0}^\infty \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n
\end{aligned}$$

■

Theorem 5.22. *If $\sum_{n=0}^\infty a_n(z-z_0)^n$ converges to $f(z)$ in the open ball $B(z_0,R)$, then the series is the Taylor series of f at z_0.*

Proof. Choose r such that $0 < r < R$, and let γ be the positive oriented circle $|z-z_0| = r$. Let

$$g_k(z) = \frac{1}{2\pi i(z-z_0)^{k+1}}$$

so that by Cauchy's integral formula

$$\int_\gamma g_k(z)\phi(z)\,\mathrm{d}z = \frac{\phi^{(k)}(z_0)}{k!}.$$

Now

$$\begin{aligned}
\int_\gamma g_k(z)(z-z_0)^n\,\mathrm{d}z &= \frac{1}{k!}\frac{\mathrm{d}^k}{\mathrm{d}z^k}(z-z_0)^n\bigg|_{z=z_0} \\
&= \begin{cases} 1, & \text{if } k=n \\ 0, & \text{otherwise} \end{cases}.
\end{aligned}$$

Thus

$$\begin{aligned}
a_k &= \sum_{n=0}^\infty a_n \int_\gamma g_k(z)(z-z_0)^n\,\mathrm{d}z \\
&= \int_\gamma g_k(z)f(z)\,\mathrm{d}z \\
&= \frac{f^{(k)}(z_0)}{k!}.
\end{aligned}$$

■

The Taylor series where $z_0 = 0$ is also called the Maclaurin series.

Example 5.2. Find the Maclaurin series of $f(z) = e^z$. We have $f^{(n)}(0) = 1$. Thus the Maclaurin series is simply

$$e^z = \sum_{n=0}^\infty \frac{z^n}{n!}.$$

◊

Example 5.3. Find the Taylor series of $f(z) = 1/z$ at $z_0 = 1$. We can use the geometric series

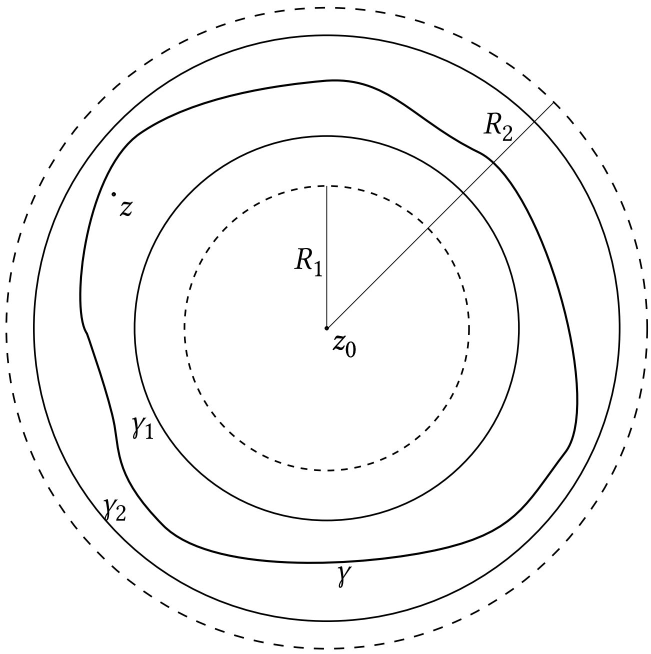
$$\begin{aligned}\frac{1}{z} &= \frac{1}{1-(1-z)} \\ &= \sum_{n=0}^{\infty}(1-z)^n\end{aligned}$$

which converges for all $|1-z| < 1$. This agrees with Taylor's theorem because $f(z)$ is not analytic at $z = 0$, so the largest ball it is analytic in is $B(1, 1)$. ♢

Theorem 5.23 (Laurent's theorem)**.** *If f is analytic in an annulus $A = \{z \mid R_1 < |z - z_0| < R_2\}$, then*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$$

Proof. For $z \in A$, let γ_1 and γ_2 be the positively oriented circles contained in A such that the γ and z are contained in the region between them. See the figure.



From example 4.5 we have the following

$$f(z) = \frac{1}{2\pi i}\left[\int_{\gamma_2} \frac{f(w)}{w-z}\,\mathrm{d}w - \int_{\gamma_1} \frac{f(w)}{w-z}\,\mathrm{d}w\right].$$

Our task is to evaluate the two path integrals.

For $w \in \{C_2\}$,

$$\begin{aligned}\frac{1}{w-z} &= \frac{1}{w-z_0}\frac{1}{1-\frac{z-z_0}{w-z_0}} \\ &= \frac{1}{w-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^n\end{aligned}$$

Thus

$$\begin{aligned}\int_{\gamma_2} \frac{f(w)}{w-z}\,\mathrm{d}w &= \frac{1}{2\pi i}\int_{\gamma_2} f(w)\sum_{n=0}^{\infty}\frac{(z-z_0)^n}{(w-z_0)^{n+1}}\,\mathrm{d}w \\ &= \sum_{n=0}^{\infty}\left[\frac{1}{2\pi i}\int_{\gamma}\frac{f(w)}{(w-z_0)^{n+1}}\,\mathrm{d}w\right](z-z_0)^n\end{aligned}$$

where the last step is due to the Cauchy-Goursat theorem. The same goes for $w \in \{C_1\}$:

$$\begin{aligned}\int_{\gamma_1} \frac{f(w)}{z-w}\,\mathrm{d}w &= \frac{1}{2\pi i}\int_{\gamma_1} f(w)\sum_{n=0}^{\infty}\frac{(w-z_0)^n}{(z-z_0)^{n+1}}\,\mathrm{d}w \\ &= \sum_{n=0}^{\infty}\left[\frac{1}{2\pi i}\int_{\gamma} f(w)(w-z_0)^n\,\mathrm{d}w\right](z-z_0)^{-n-1} \\ &= \sum_{n=-1}^{-\infty}\left[\frac{1}{2\pi i}\int_{\gamma}\frac{f(w)}{(w-z_0)^{n+1}}\,\mathrm{d}w\right](z-z_0)^n\end{aligned}$$

■

Note that if f is analytic in $B(z_0, R_2)$, then since $f(w)(w-z_0)^{n+1}$ is analytic, the integrals for the negative indices will all vanish. In this case the Laurent series will reduce to the Taylor series.

Definition 5.10 (Principle and analytic parts)**.** The terms in the Laurent series with $n \geq 0$ are collectively called the *analytic part*, while the terms with $n < 0$ are collectively called the *principle part*. □

6 Residues and poles

6.1 Isolated singularities

Definition 6.1 (Singluar points)**.** A point z_0 is a *singular point* of a function f if f is not analytic at z_0 but is analytic at some point in $B(z_0, \epsilon)$ for all $\epsilon > 0$. We say that a singular point z_0 is *isolated* if there exists $R > 0$ such that f is analytic in $B(z_0, R) \setminus \{z_0\}$. □

Example 6.1. $\operatorname{Log} z$ is analytic in $\mathbb{C} \setminus (-\infty, 0]$. Every point in $(-\infty, 0]$ is a singular point of $\operatorname{Log} z$, but they are not isolated singularities. ♢

Example 6.2. Let $f(z) = 1/\sin(\pi/z)$. $\sin(\pi/z) = 0$ iff $z = 1/n$ for $n \in \mathbb{Z}^+$. So the singular points of f are $\{0\} \cup \{1/n \mid n \in \mathbb{Z}^+\}$. The singularities at $1/n$ are isolated. However, 0 is not an isolated singularity since for every R, we can always find n such that $1/n \in B(0, R)$. ♢

Definition 6.2 (Residues)**.** The residue of f at an isolated singularity z_0 is the $n = -1$ coefficient of its Laurent series expansion, or in other words

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{2\pi i}\int_{\gamma} f(z)\,\mathrm{d}z$$

where γ is any positively oriented simple closed contour around z_0 in $B(z_0, R) \setminus \{z_0\}$. □

Example 6.3. We want to evaluate $\int_\gamma ze^{4/z}\,\mathrm{d}z$ where γ is the unit circle. We have

$$\begin{aligned} ze^{4/z} &= z\sum_{n=0}^{\infty}\frac{(4/z)^n}{n!} \\ &= \sum_{n=0}^{\infty}\frac{4^n}{n!z^{n-1}}. \end{aligned}$$

Thus $\mathrm{Res}_{z=0}\, f$ is the coefficient of $1/z$, which is 8. Then

$$\int_\gamma ze^{4/z}\,\mathrm{d}z = 2\pi i \operatorname*{Res}_{z=0} ze^{4/z} = 16\pi i.$$

◊

Definition 6.3 (Removable singularities)**.** Let f have an isolated singular point at z_0. If the principle part of the Laurent series of f around $z = z_0$ is 0, then we say that z_0 is a *removable* singularity. □

For removable singularities the Laurent series reduces to a power series and the residue there is 0. Note that is is not necessarily a Taylor series since f is still not analytic in the whole ball. However we can make f analytic in the whole ball if we set $f(z_0) = a_0$. This also explains why such singularities are called “removable”.

Example 6.4. Consider $f(z) = \frac{\sin z}{z}$.

$$\begin{aligned} \frac{\sin z}{z} &= \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right) \\ &= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots \end{aligned}$$

so $z = 0$ is a removable singularity. If we redefine $f(0) = 1$, then $f(z)$ is equal to the above convergent Taylor series. Thus it becomes analytic at $z = 0$. ◊

Definition 6.4 (Essential singularity)**.** Let f have an isolated singular point at z_0. If the principle part of the Laurent series of f around $z = z_0$ has infinitely many non-zero terms then we say that z_0 is an *essential* singularity. □

Example 6.5. For all z,

$$\begin{aligned} e^{1/z} &= \sum_{n=0}^{\infty}\frac{(1/z)^n}{n!z^n} \\ &= 1 + \sum_{n=1}^{\infty}\frac{1}{n!z^n} \end{aligned}$$

so $z = 0$ is an essential singularity. ◊

Definition 6.5 (Poles)**.** Let f have an isolated singular point at z_0. Consider the Laurent series of f around $z = z_0$. If there is $N \in \mathbb{N}$ such that the coefficients $a_{-n} = 0$ for all $n > m$, then we say z_0 is a *pole*. Furthermore, we call the smallest possible value of N the *order* of the pole. □

We sometimes call poles of order 1 *simple poles.*

Example 6.6. For all z,

$$\begin{aligned}\frac{e^z}{z^2} &= \frac{1}{z^2}\sum_{n=0}^{\infty}\frac{z^n}{n!} \\ &= \frac{1}{z^2} + \frac{1}{z} + \frac{1}{2} + \cdots\end{aligned}$$

so $z = 0$ is a pole of order 2. ◊

Theorem 6.1. *A function f has a pole of order m at z_0 iff there exists $R > 0$ such that for all $z \in B(z_0, R) \setminus \{z_0\}$,*

$$f(z) = \frac{\phi(z)}{(z - z_0)^m}$$

where ϕ is analytic and $\phi(z_0) \neq 0$.

Proof.

($\Longrightarrow$) Since f has a pole of order m at z_0, there exists R such that that for all $z \in B(z_0, R) \setminus \{z_0\}$,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{m} \frac{a_{-n}}{(z - z_0)^n}.$$

Now consider if we multiply the series throughout by $(z - z_0)^m$:

$$\begin{aligned}\phi(z) &= \sum_{n=0}^{\infty} a_n (z - z_0)^{n+m} + a_{-1}(z - z_0)^{m-1} + \cdots + a_{-(m-1)}(z - z_0) + a_{-m} \\ &= \begin{cases} f(z)(z - z_0)^m, & \text{if } z \neq z_0 \\ a_{-m}, & \text{otherwise} \end{cases}.\end{aligned}$$

This is an analytic function in $B(z_0, R)$ (theorem 5.20).

($\Longleftarrow$). Since ϕ is analytic at z_0, it has a Taylor expansion

$$\phi(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

for all $z \in B(z_0, R)$, for some $R > 0$. Thus

$$\begin{aligned}f(z) &= \frac{\phi(z)}{(z - z_0)^m} \\ &= \frac{c_0}{(z - z_0)^m} + \cdots + \frac{c_{m-1}}{z - z_0} + \cdots\end{aligned}$$

and as $c_0 = \phi(z_0) \neq$ by definition, f has a pole of order m at $z = z_0$. ■

Corollary 6.0.1. *A function f has a pole at $z = z_0$, iff $\lim_{z \to z_0} f(z) = \infty$.*

Proof. Equivalently,

$$\begin{aligned}\lim_{z \to z_0} \frac{1}{f(z)} &= \lim_{z \to z_0} \frac{(z - z_0)^m}{\phi(z)} \\ &= \frac{(z_0 - z_0)^m}{\phi(z_0)} \\ &= 0.\end{aligned}$$

■

Theorem 6.2. *Suppose that a function f has an isolated singularity at $z = z_0$. Then if $z = z_0$ is a removable singularity, f is bounded in a deleted neighbourhood of z_0.*

Proof. The function f is analytic in $B(z_0, R)$ for some R if we define $f(z_0)$ correctly. Then f is continuous in the closed ball $\overline{B(z_0, r)}$ for all $r < R$. Since it is closed f is bounded there as well. Then it must also be bounded on the deleted neighbourhood $\{z \mid 0 < |z - z_0| < r\} \subset \overline{B(z_0, r)}$. ■

Theorem 6.3 (Reimann's theorem)**.** *Suppose that a function f is bounded and analytic in some deleted neighbourhood $\{z \mid 0 < |z - z_0| < R\}$ of z_0. If f has a singularity at z_0, then it is a removable singularity.*

Proof. If f is not analytic at z_0, then it must be an isolated singularity. So we can represent f by a Laurent series in the deleted neighbourhood. Let γ be the positively oriented circle $|z - z_0| = r$ where $r < R$. Since f is bounded, $|f(z)| \leq M$ for some M. Then by the ML-inequality, the coefficients of the principal part of the Laurent series ($n < 0$) are

$$\begin{aligned} |a_n| &= \left| \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)^{n+1}} \, \mathrm{d}z \right| \\ &\leq \frac{1}{2\pi} \frac{M}{r^{n+1}} 2\pi r \\ &= Mr^{-n}. \end{aligned}$$

Since we can choose r to be arbitrarily small, we can conclude that the principal of the Laurent series is 0. ■

Theorem 6.4 (Picard's theorem)**.** *If f has an essential singularity at $z = z_0$, then in any open neighbourhood of z_0, f assumes every finite value, with one possible exception, for an infinite number of times.*

Theorem 6.5. *If f has a pole of order m at z_0, then*

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}} (z - z_0)^m f(z).$$

Proof. If f has a pole of order m at z_0, then there exists $R > 0$ such that for all z where $0 < |z - z_0| < R$,

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{a_{-1}}{z - z_0} + \cdots + \frac{a_{-m}}{(z - z_0)^m} \\ (z - z_0)^m f(z) &= \sum_{n=0}^{\infty} a_n (z - z_0)^{n+m} + a_{-1}(z - z_0)^{m-1} + \cdots + a_{-m} \\ \frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}} (z - z_0)^m f(z) &= \sum_{n=0}^{\infty} \left(\prod_{k=n+2}^{n+m} k \right) a_n (z - z_0)^{n+1} + (m-1)! a_{-1} \\ \lim_{z \to z_0} \frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}} (z - z_0)^m f(z) &= (m-1)! a_{-1}. \end{aligned}$$

■

6.2 Poles and zeroes

Definition 6.6 (Zeroes)**.** A point z_0 is called a *zero* of f if $f(z_0) = 0$. If $f(z_0) = \cdots = f^{(m-1)}(z_0) = 0$ but $f^{(m)}(z_0) \neq 0$, then we say that z_0 is a zero of *order m*. □

We often call a zero of order 1 a simple zero.

Example 6.7. The function $f(z) = z(e^z - 1)$ has zeroes at $z = 2n\pi i$ with $n \in \mathbb{Z}$. First consider the zero at $z = 0$.

$$f'(z) = (z+1)e^z - 1 \qquad f''(z) = (z+2)e^z$$
$$f'(0) = 0 \qquad f''(0) = 2$$

so the zero at $z = 0$ is of order 2. For the other zeroes, $f'(2n\pi i) \neq 0$ for $n \neq 0$ so they are simple zeroes. ♢

Theorem 6.6. *Let f be analytic at z_0. Then f has a zero of order m at z_0 iff $f(z) = (z-z_0)^m g(z)$, where g is analytic at z_0 and $g(z_0) \neq 0$.*

Proof.

($\Longrightarrow$) Since f is analytic at $z = z_0$, it has a Taylor series for all $z \in B(z_0, R)$ for some R. However since the first $m-1$ derivatives at $z = z_0$ are all 0, the first $m-1$ coefficients are 0 as well. Thus

$$\begin{aligned} f(z) &= \sum_{n=m}^{\infty} a_n (z-z_0)^n \\ &= (z-z_0)^m \sum_{n=m}^{\infty} a_n (z-z_0)^{n-m} \\ &= (z-z_0)^m g(z) \end{aligned}$$

We define $g(z)$ be the summation term. It is represented by a convergent power series in $B(z_0, R)$, so it is also analytic at z_0. Furthermore $g(z_0) = a_m \neq 0$.

($\Longleftarrow$) Since g is analytic at $z = z_0$, it has a Taylor series for all $z \in B(z_0, R)$ for some R.

$$g(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n.$$

Furthermore, $c_0 = g(z_0) \neq 0$. Thus

$$\begin{aligned} f(z) &= (z-z_0)^m g(z) \\ &= \sum_{n=0}^{\infty} c_n (z-z_0)^{n+m} \end{aligned}$$

Thus it is clear that

$$f(z_0) = \cdots = f^{(m-1)}(z_0) = 0 \qquad f^{(m)}(z_0) = m! c_0 \neq 0.$$

■

Theorem 6.7. *Let p and q be analytic at z_0 and suppose $p(z_0) \neq 0$. Then if q has a zero of order m at z_0, the function $f(z) = p(z)/q(z)$ has a pole of order m at z_0.*

Proof. There exists $R > 0$ such that for all $z \in B(z_0, R)$,

$$q(z) = (z - z_0)^m g(z)$$

where g is analytic at z_0 and $g(z_0) \neq 0$. Then

$$f(z) = \frac{p(z)}{(z - z_0)^m g(z)}$$

where $p(z)/g(z)$ is analytic and non-zero at $z = z_0$. Thus f has a pole of order m at $z = z_0$. ■

Example 6.8. Consider the function $f(z) = \frac{e^z}{z(e^z-1)}$. From example 6.7, we know that $z(e^z - 1)$ has a zero of order 2 at $z = 0$ and zeroes of order 1 at $z = 2n\pi i$ for $n \in \mathbb{Z} \setminus \{0\}$. Furthermore $e^z \neq 0$ at these values. Thus f has a double pole at $z = 0$ and simple poles at $z = 2n\pi i$ for $n \in \mathbb{Z} \setminus \{0\}$. ♢

Corollary 6.0.2. *If p and q are analytic at z_0 and $p(z_0) \neq 0$ and q has a simple zero at z_0, then $f(z) = p(z)/q(z)$ has a simple pole at z_0, and furthermore*

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{p(z_0)}{q'(z_0)}.$$

Proof. Since f has a simple pole at z_0 and $q(z_0) = 0$,

$$\begin{aligned}
\operatorname*{Res}_{z=z_0} f(z) &= \lim_{z \to z_0} (z - z_0) f(z) \\
&= \lim_{z \to z_0} \frac{p(z)}{\frac{q(z)-q(z_0)}{z-z_0}} \\
&= \frac{p(z_0)}{q'(z_0)}.
\end{aligned}$$

■

We now consider the general case for $f(z) = p(z)/q(z)$. By the quotient rule,

$$f'(z) = \frac{q(z)p'(z) - p(z)q'(z)}{q^2(z)}$$

exists provided $q(z) \neq 0$. Suppose q has a zero of order n at $z = z_0$. If $p(z_0) \neq 0$, then f has a pole of order n at $z = z_0$.

What if instead $p(z_0) = 0$? Suppose p has a zero of order m at $z = z_0$. Then there exists analytic functions p_1 and q_1, with $p_1(z_0) \neq 0$ and $q_1(z_0) \neq 0$, such that

$$\begin{aligned}
f(z) &= \frac{(z - z_0)^m p_1(z)}{(z - z_0)^n q_1(z)} \\
&= (z - z_0)^{m-n} \phi(z)
\end{aligned}$$

where $\phi(z) = p_1(z)/q_1(z)$ and $\phi(z_0) \neq 0$. If $m \geq n$, then

$$\lim_{z \to z_0} f(z) = \begin{cases} 0, & \text{if } m > n \\ \phi(z_0), & \text{otherwise} \end{cases}.$$

In particular, f is bounded. Consequently, f has a removable singularity at $z = z_0$. To be precise, $f(z)$ has a zero of order $m - n$. If instead $m < n$, then

$$f(z) = \frac{\phi(z)}{(z - z_0)^{n-m}}$$

so f has a pole of order $n - m$.

Theorem 6.8 (Cauchy's residue theorem). *If Γ is a positively oriented simple closed contour and f is analytic inside and on Γ except for a finite number of singular points $z_1, \dots, z_k$, then*

$$\int_\Gamma f(z)\,\mathrm{d}z = 2\pi i \sum_{n=1}^{k} \operatorname*{Res}_{z=z_n} f(z).$$

Proof. Let the points $z_1, \dots, z_k$ be the centres of positively oriented circles $\gamma_1, \dots, \gamma_k$ which are interior to Γ and are small enough such that they do not overlap one another. The circles, together with the contour Γ, form a closed region whose interior is a multiply connected domain consisting of the points inside Γ but outside all γ_k. f is analytic inside this region. Using the Cauchy-Goursat theorem for multiply connected domains,

$$\int_\Gamma f(z)\,\mathrm{d}z - \sum_{n=1}^{k} \int_{\gamma_n} f(z)\,\mathrm{d}z = 0$$

which leads directly to the desired result since

$$\int_{\gamma_n} f(z)\,\mathrm{d}z = 2\pi i \operatorname*{Res}_{z=z_n} f(z).$$

■

6.3 Applications

Definition 6.7 (Improper integrals). Let $f : [0, \infty) \to \mathbb{R}$. The *improper integral* of f over $[0, \infty)$ is defined by

$$\int_0^\infty f(x)\,\mathrm{d}x = \lim_{R\to\infty} \int_0^R f(x)\,\mathrm{d}x$$

and we say that the integral converges provided the limit exists. □

The same definition can be made for integrals over $(-\infty, 0]$. Integrals over $(-\infty, \infty)$ are the sum of these two types of integrals, i.e. $\int_{-\infty}^{\infty} = \int_{-\infty}^{0} + \int_0^\infty$.

Definition 6.8 (Cauchy principal value). The *Cauchy principal value* of $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$ is defined as

$$\text{p.v.} \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,\mathrm{d}x$$

and we say that it converges provided the limit exists. □

It should be noted that the principal value is different from our original definition of the indefinite integral. If $\int_{-\infty}^{\infty}$ exists, then p.v. $\int_{-\infty}^{\infty} = \int_{-\infty}^{\infty}$. However the converse is not necessarily true.

Example 6.9. Consider

$$\begin{aligned}\text{p.v.} \int_{-\infty}^{\infty} x\,\mathrm{d}x &= \lim_{R\to\infty} \int_{-R}^{R} x\,\mathrm{d}x \\ &= 0.\end{aligned}$$

On the other hand,

$$\lim_{R\to\infty}\int_0^R x\,\mathrm{d}x = \infty.$$

So clearly $\int_{-\infty}^{\infty} x\,\mathrm{d}x$ does not converge. $\lozenge$

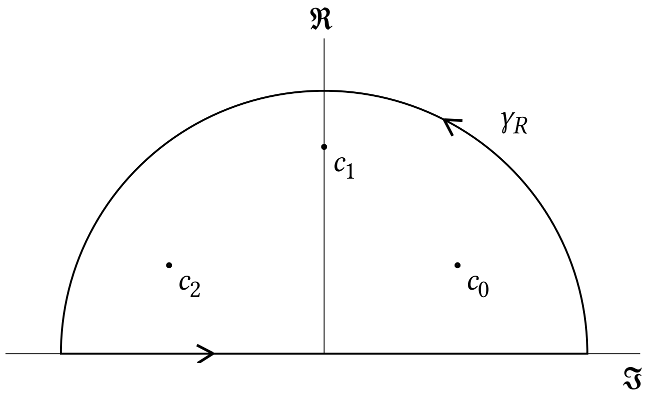
Theorem 6.9. *Let f be an even function, i.e. $f(-x) = f(x)$. If* p.v. $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$ *converges, then so does* $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$.

Proof. We have

$$\int_{-R}^{0} f(x)\,\mathrm{d}x = \int_{R}^{0} f(-x)\,\mathrm{d}(-x) = \int_{0}^{R} f(x)\,\mathrm{d}x\,.$$

Therefore, if p.v. $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \lim_{R\to\infty}\int_{-R}^{R} f(x)\,\mathrm{d}x$ converges, then both $\lim_{R\to\infty}\int_{-R}^{0} f(x)\,\mathrm{d}x$ and $\lim_{R\to\infty}\int_{0}^{R} f(x)\,\mathrm{d}x$ must exist. ■

Example 6.10. Let us try and evaluate $\int_{-\infty}^{\infty}\frac{x^2}{x^6+1}\,\mathrm{d}x$. Note that the singular points of f are at $c_n = \exp\left(\frac{(2n+1)\pi i}{6}\right)$ and they are all simple poles. Let Γ_R be the positively oriented semicircle of radius R containing c_0, c_1, and c_0. Denote the arc as γ_R.



By Cauchy's residue theorem,

$$\begin{aligned} 2\pi i\sum_{n=0}^{2}\operatorname*{Res}_{z=c_n} f(z) &= \int_\Gamma f(z)\,\mathrm{d}z \\ &= \int_{\gamma_R} f(z)\,\mathrm{d}z + \int_{-R}^{R} f(x)\,\mathrm{d}x\,. \end{aligned}$$

Recall we have a formula for this specific kind of poles (corollary 6.0.2):

$$\operatorname*{Res}_{z=c_n} f(z) = \frac{c_k^2}{6c_k^5}$$

so

$$\int_{-R}^{R} f(x)\,\mathrm{d}x = \frac{\pi}{3} - \int_{\gamma_R} f(z)\,\mathrm{d}z\,.$$

Now for all $z \in \{\gamma_R\}$, we have $|z| = R$, such that

$$\begin{aligned} |f(z)| &\leq \frac{|z|^2}{\left||z|^6 - 1\right|} \\ &= \frac{R^2}{R^6 - 1}. \end{aligned}$$

Thus by the ML-inequality

$$\left| \int_{\gamma_R} f(z) \, \mathrm{d}z \right| \leq \frac{R^2}{R^6 - 1} \pi R.$$

Now when we make $R \to \infty$, this integral goes to 0. Therefore we conclude that in fact

$$\int_{-\infty}^{\infty} \frac{x^2}{x^6 + 1} \, \mathrm{d}x = \frac{\pi}{3}.$$

$\diamondsuit$

Example 6.11. Let us try and evaluate $\int_{-\infty}^{\infty} \frac{\cos 3x}{(x^2+1)^2} \, \mathrm{d}x$. Instead consider $f(z) = \frac{\exp(i3z)}{(z^2+1)^2}$ first. It has double poles at $z = \pm i$. Let Γ_R be the positively oriented semicircle of radius R containing $z = i$, and let γ_R be its arc. Firstly

$$\begin{aligned} \operatorname*{Res}_{z=i} f(z) &= \lim_{z \to i} \frac{\mathrm{d}}{\mathrm{d}z} (z - i)^2 \frac{e^{i3z}}{(z^2 + 1)^2} \\ &= \lim_{z \to i} \frac{\mathrm{d}}{\mathrm{d}z} \frac{e^{i3z}}{(z + i)^2} \\ &= \frac{1}{e^3 i}. \end{aligned}$$

Next apply Cauchy's residue theorem

$$\begin{aligned} \int_{-R}^{R} \frac{e^{i3x}}{(x^2 + 1)^2} \, \mathrm{d}x &= \frac{2\pi}{e^3} - \int_{\gamma_R} f(z) \, \mathrm{d}z \\ \int_{-R}^{R} \frac{\cos 3x}{(x^2 + 1)^2} \, \mathrm{d}x &= \Re\left\{ \frac{2\pi}{e^3} - \int_{\gamma_R} f(z) \, \mathrm{d}z \right\}. \end{aligned}$$

Now for $z \in \{\gamma_R\}$, we have $|z| = R$.

$$\begin{aligned} |f(z)| &= \frac{e^{3y}}{(z^2 + 1)^2} \\ &\leq \frac{e^{i3y}}{(|z|^2 + 1)^2} \\ &\leq \frac{1}{(R^2 + 1)^2}. \end{aligned}$$

The last step arises from noting that $y \geq 0$ along the arc. Now by the ML-inequality

$$\begin{aligned} \left| \Re \int_{\gamma_R} f(z) \, \mathrm{d}z \right| &\leq \left| \int_{\gamma_R} f(z) \, \mathrm{d}z \right| \\ &\leq \frac{1}{(R^2 - 1)^2} \pi R \end{aligned}$$

which goes to 0 as $R \to \infty$. Thus we conclude that in fact

$$\int_{-\infty}^{\infty} \frac{\cos 3x}{(x^2+1)^2}\,\mathrm{d}x = \frac{2\pi}{e^3}.$$

◊