1 Calculus

1.1 General Leibniz Rule

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}[f(x)g(x)] = \sum_{r=0}^{n} f^{(r)}(x)g^{(n-r)}(x)$$

1.2 Leibniz Rule

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{u(x)}^{v(x)} f(x,t)\,\mathrm{d}t = f(x,v(x))\frac{\mathrm{d}v(x)}{\mathrm{d}x} - f(x,u(x))\frac{\mathrm{d}u(x)}{\mathrm{d}x} + \int_{u(x)}^{v(x)} \frac{\partial f(x.t)}{\partial x}\,\mathrm{d}t$$

1.3 Jacobian

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \partial_u x & \partial_v x & \partial_w x \\ \partial_u y & \partial_v y & \partial_w y \\ \partial_u z & \partial_v z & \partial_w z \end{vmatrix}$$

$$\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w$$

$$J_{\mathbf{xz}} = J_{\mathbf{xy}} J_{\mathbf{yz}}$$

$$J_{\mathbf{xx}} = \mathbf{I}$$

1.4 Fourier Series

Generalized Fourier series over an orthonormal basis in $[a,b]$ with $2L = |b-a|$

$$f(x) = \sum_{n=0}^{\infty} c_n\phi_n(x) \qquad c_n = \frac{\langle\phi_n|f\rangle}{\|\phi_n\|^2}$$

$$\phi = \sin\left(\frac{n\pi x}{L}\right) \text{ or } \cos\left(\frac{n\pi x}{L}\right) \implies \|\phi\|^2 = \frac{L}{2}$$

$$\phi = \exp\left(i\frac{n\pi x}{L}\right) \implies \|\phi\|^2 = 2L$$

Parseval's Theorem

$$\left\langle f(x)^2\right\rangle = \frac{1}{2L}\int_{-L}^{L} f(x)^2\,\mathrm{d}x = \left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2) = \sum_{n=-\infty}^{\infty}|c_n|^2$$

For BVP $y''(x) + cy = f(x), 0 \le x \le L$:

$$y(0) = y(L) = 0: \quad y(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$y'(0) = y'(L) = 0: \quad y(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

1.5 Fourier Transform

$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx} f(x)\,\mathrm{d}x$, $f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\tilde{f}(k)\,\mathrm{d}k$.

$$\mathcal{F}\left\{\frac{\mathrm{d}^n f(x)}{\mathrm{d}x^n}\right\} = (ik)^n\mathcal{F}\{f(x)\}$$

$$\mathcal{F}\left\{\int_{-\infty}^{x} f(u)\,\mathrm{d}u\right\} = \frac{1}{ik}\tilde{f}(k)$$

$$\mathcal{F}\{f(x)\} = \tilde{f}(k) \implies \mathcal{F}\left\{\tilde{f}(x)\right\} = f(-k)$$

$$\mathcal{F}\{f(x+a)\} = e^{ika}\tilde{f}(k) \iff \mathcal{F}^{-1}\left\{e^{ika}\tilde{f}(k)\right\} = f(x+a)$$

$$\mathcal{F}^{-1}\left\{\tilde{f}(x+a)\right\} = e^{-ixa}\tilde{f}(x) \iff \mathcal{F}\left\{e^{-ixa}f(a)\right\} = \tilde{f}(k+a)$$

$$\mathcal{F}\{f(ax)\} = \frac{1}{|a|}\tilde{f}\left(\frac{k}{a}\right) \qquad \mathcal{F}^{-1}\left\{\tilde{f}(ak)\right\} = \frac{1}{|a|}f\left(\frac{x}{a}\right)$$

$$\int_{-\infty}^{\infty} f^*(x)g(x)\,\mathrm{d}x = \int_{-\infty}^{\infty}\tilde{f}^*(k)\tilde{g}(k)\,\mathrm{d}x$$

$$\mathcal{F}\{f(x)*g(x)\} = \sqrt{2\pi}\tilde{f}(k)\tilde{g}(k) \qquad \mathcal{F}\{f(x)g(x)\} = \frac{1}{\sqrt{2\pi}}\tilde{f}(k)*\tilde{g}(k)$$

1.6 Dirac Delta Distribution

$\delta(x-x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-x')}\,\mathrm{d}k$. It is even. $\delta[a(x-x')] = \frac{1}{|a|}\delta(x-x')$. $f(x) = \int_{-\infty}^{\infty} f(x')\delta(x-x')\,\mathrm{d}x'$. $\int_{-\infty}^{\infty} f(x)\delta^{(n)}(x-x')\,\mathrm{d}x = (-1)^n f^{(x)}(x')$ 3D Dirac distribution – remember the Jacobian!

1.7 Laplace Transform

$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)\,\mathrm{d}t\,, \mathrm{Re}\{s\} > 0.$

$$\mathcal{L}\left\{\frac{\mathrm{d}^n f(t)}{\mathrm{d}t^n}\right\} = s^n\overline{f}(s) - \sum_{r=0}^{n-1} s^{n-1-r}f^{(r)}(0)$$

$$\mathcal{L}\left\{\int_0^t f(u)\,\mathrm{d}u\right\} = \frac{1}{s}\overline{f}(s)$$

$$\frac{\mathrm{d}^n}{\mathrm{d}s^n}\mathcal{L}\{f(t)\} = \mathcal{L}\{(-t)^n f(t)\}$$

$$\int_s^\infty \mathcal{L}\{f(t)\}\,\mathrm{d}s' = \mathcal{L}\left\{\frac{f(t)}{t}\right\}$$

$$\mathcal{L}\{f(at)\} = \frac{1}{a}\overline{f}(\frac{s}{a})$$

$$\mathcal{L}\left\{e^{at}f(t)\right\} = \overline{f}(s-a) \qquad \mathcal{L}\{H(t-a)f(t-a)\} = e^{-sa}\overline{f}(s)$$

$$\mathcal{L}\{f(t)*g(t)\} = \overline{f}(s)\overline{g}(s) \qquad \mathcal{L}^{-1}\left\{\overline{f}(s)\overline{g}(s)\right\} = f(t)*g(t)$$

$$\mathcal{L}\left\{e^{at}\right\} = \frac{1}{s-a} \qquad \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}\{\sin\omega t\} = \frac{\omega}{s^2+\omega^2} \qquad \mathcal{L}\{\cos\omega t\} = \frac{s}{s^2+\omega^2}$$

$$\mathcal{L}\{\sinh\omega t\} = \frac{\omega}{s^2-\omega^2} \qquad \mathcal{L}\{\cosh\omega t\} = \frac{s}{s^2-\omega^2}$$

1.8 ODEs

$W(x) = \begin{vmatrix} y_1(x) & y_2(x) & \dots & y_N(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(N-1)}(x) & y_2^{(N-1)}(x) & \dots & y_N^{(N-1)}(x)\end{vmatrix}$. $W(x) = 0 \implies$ linearly independence in that region.

1.8.1 Solution with power series

About an ordinary point, $x = x_0 = 0$ try just using power series. A change of variables $X = x - x_0$ can be made to solve around $X = X_0 = 0$.

For regular singular points we can try Frobenius' method of $y = (x-x_0)^\sigma\sum_{n=0}^\infty(x-x_0)^n, a_0 \neq 0$.

1. Write in the form of $x^2p_2(x)y'' + xp_1(x)y' + p_0y = 0$,
2. Write down $\sum_{n=0}^\infty[p_2(x)(n+\sigma)(n+\sigma-1) + p_1(x)(n+\sigma) + p_0(x)]a_nx^n = 0$
3. If x is ordinary then $\sigma_2 = 0$ gives the general solution.
4. If the σ s differ by a non-integer then two linearly independent solutions are obtained.
5. Else if they differ by a non-zero integer then the smaller one may or may not be linearly independent. $y_2 = Cy_1(x)\ln|x| + x^{\sigma_2}\sum_{n=0}^\infty, C = 0 \implies b_nx^n, b_{\sigma_1-\sigma_2} = 0$.
6. Else (they are equal) then $y_2 = y_1(x)\ln|x| + x^{\sigma_2}\sum_{n=1}^\infty b_nx^n$.
7. Generally $y_2 = y_1(x)\int^x \frac{1}{y_1^2(\xi)}\exp\left\{-\int^\xi \frac{p_1(\xi')}{p_2(\xi')}\,\mathrm{d}\xi'\right\}\mathrm{d}\xi$

1.8.2 Particular Integral

$$y_p(x) = -\int^x \frac{q(\xi)}{p_2(\xi)}\frac{y_2(\xi)y_1(x) - y_1(\xi)y_2(x)}{y_1(\xi)y_2'(\xi) - y_2(\xi)y_1'(\xi)}\,\mathrm{d}\xi$$

1.9 Sturm-Liouville

1.9.1 Sturm-Liouville Equations

$\mathcal{L} = \frac{\mathrm{d}}{\mathrm{d}x}\left[p_2(x)\frac{\mathrm{d}}{\mathrm{d}x}\right] + p_0(x)$, for $\frac{\mathrm{d}}{\mathrm{d}x}\left[\mu(x)p_2(x)\frac{\mathrm{d}y(x)}{\mathrm{d}x}\right] + \mu(x)p_0(x)y(x) - \lambda\mu(x)w(x)y(x) = 0$. More generally:

If $p_1 = \frac{\mathrm{d}p_2(x)}{\mathrm{d}x}$, $\mu = 1$, otherwise use

$$\mu(x) = \exp\left\{\int^x \frac{p_1(\xi) - p_2'(\xi)}{p_2(\xi)}\,\mathrm{d}\xi\right\}$$

$$y(x) = \int_a^b G(x,x')q(x')\,\mathrm{d}x'$$

1.9.2 Eigenfunctions

1. Write down $\mathcal{L}\phi(x) = \lambda\mu(x)\phi(x)$. Solve and obtain a spectrum.
2. Normalize all ϕ.
3. $G(x,x') = \sum_{n=0}^\infty \frac{1}{\lambda_n}\hat{\phi}_n^*(x')\hat{\phi}_n(x) = G^*(x',x)$

1.10 Green's Function

1.10.1 Dirac delta method (impulse)

1. Solve homo equation $\mathcal{L}y_{1,2}(x) = 0$.
2. Do

$$G(x,x') = \begin{cases} \alpha_1(x')y_1(x) + \alpha_2(x')y_2(x), a \le x \le x' \\ \beta_1(x')y_1(x) + \beta_2(x')y_2(x), x' \le x \le b \end{cases}$$

3. Impose boundary conditions:

$$\begin{cases} y(a) = 0 \implies G(a,x') = 0 \\ y(b) = 0 \implies G(b,x') = 0 \end{cases} \quad \begin{cases} y'(a) = 0 \implies G'(a,x') = 0 \\ y'(b) = 0 \implies G'(b,x') = 0 \end{cases}$$

$$\begin{cases} y(a) = 0 \implies G(a,x') = 0 \\ y'(b) = 0 \implies G'(b,x') = 0 \end{cases} \quad \begin{cases} y'(a) = 0 \implies G'(a,x') = 0 \\ y(b) = 0 \implies G(b,x') = 0 \end{cases}$$

4. Impose continuity/discontinuity conditions at $x = x'$:

$$\begin{cases} \lim_{\epsilon\to 0^+} \left[G(x,x')\big|_{x=x'+\epsilon} - G(x,x')\big|_{x=x'-\epsilon} \right] = 0 \\ \lim_{\epsilon\to 0^+} \left[G'(x,x')\big|_{x=x'+\epsilon} - G'(x,x')\big|_{x=x'-\epsilon} \right] = -\frac{1}{p_2(x')} \end{cases}$$

2 Vector calculus

2.1 Identities

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$$
$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$$

$$\begin{aligned}
\boldsymbol{\nabla} &= \Phi(\boldsymbol{\nabla}\Psi) + (\boldsymbol{\nabla}\Phi)\Psi \\
\boldsymbol{\nabla}(\mathbf{A}\cdot\mathbf{B}) &= \mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{B}) + \mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{A}) + (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B} + (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A} \\
\boldsymbol{\nabla}\cdot(\Phi\mathbf{A}) &= \Phi(\boldsymbol{\nabla}\cdot\mathbf{A}) + \mathbf{A}\cdot\boldsymbol{\nabla}\Phi \\
\boldsymbol{\nabla}\cdot(\mathbf{A}\times\mathbf{B}) &= \mathbf{B}\cdot(\boldsymbol{\nabla}\times\mathbf{A}) - \mathbf{A}\cdot(\boldsymbol{\nabla}\times\mathbf{B}) \\
\boldsymbol{\nabla}\times(\Phi\mathbf{A}) &= \Phi(\boldsymbol{\nabla}\times\mathbf{A}) + \boldsymbol{\nabla}\Phi\times\mathbf{A} \\
\boldsymbol{\nabla}\times(\mathbf{A}\times\mathbf{B}) &= \mathbf{A}(\boldsymbol{\nabla}\cdot\mathbf{B}) - \mathbf{B}(\boldsymbol{\nabla}\cdot\mathbf{A}) + (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A} - (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B} \\
\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A}) &= \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A}) - \nabla^2\mathbf{A}
\end{aligned}$$

$$\int_C \nabla\Phi\cdot \mathrm{d}r = \Phi(P_2) - \Phi(P_1)$$
$$\iiint_V \nabla\cdot\mathbf{A}\,\mathrm{d}V = \oiint_S \mathbf{A}\cdot\mathrm{d}\vec{a}$$
$$\oiint_S (\nabla\times\mathbf{A})\cdot\mathrm{d}\vec{a} = \oint_C \mathbf{A}\times\mathrm{d}\vec{r}$$

2.2 Vector Integration

$$\mathrm{d}s = \sqrt{\frac{\mathrm{d}\vec{r}}{\mathrm{d}u}\cdot\frac{\mathrm{d}\vec{r}}{\mathrm{d}u}}\,\mathrm{d}u \quad \mathrm{d}\mathbf{a} = \pm\frac{\partial\mathbf{r}}{\partial u}\times\frac{\partial\mathbf{r}}{\partial v}\,\mathrm{d}u\,\mathrm{d}v \quad \mathrm{d}\mathbf{a} = \frac{\boldsymbol{\nabla}F\,\mathrm{d}x\,\mathrm{d}y}{\partial F/\partial z}\bigg|_S$$

2.3 Curvilinear Coordinates

$h_i = \left|\frac{\partial\mathbf{r}}{\partial q_i}\right|$

$$\begin{aligned}
\nabla A &= \sum_i^3 \frac{1}{h_i}\frac{\partial\Phi}{\partial u_i}\hat{e}_i' \\
\nabla\cdot\mathbf{A} &= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3\mathbf{A}_1) + \frac{\partial}{\partial u_2}(h_3h_1\mathbf{A}_2) + \frac{\partial}{\partial u_3}(h_2h_1\mathbf{A}_3)\right] \\
\nabla^2\Phi &= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\Phi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\Phi}{\partial u_2}\right)\right. \\
&\quad \left. + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\Phi}{\partial u_3}\right)\right] \\
\nabla\times\mathbf{A} &= \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{e}_1' & h_2\hat{e}_2' & h_3\hat{e}_3' \\ \partial_{u_1} & \partial_{u_2} & \partial_{u_3} \\ h_1\mathbf{A}_1 & h_2\mathbf{A}_2 & h_3\mathbf{A}_3 \end{vmatrix}
\end{aligned}$$

3 Misc

3.1 Binomial Expansion

$$(1+x)^N = \sum_{n=0}^N \binom{N}{n} x^n \quad \binom{N}{n} = \frac{\prod_{i=0}^{\text{n times}}(N-i)}{n!}, \quad |x| < 1$$

3.2 MF15

$$\sin^2\theta = \frac{1-\cos 2\theta}{2} \qquad \cos^2\theta = \frac{1+\cos 2\theta}{2}$$
$$\sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4} \qquad \cos^3\theta = \frac{3\cos\theta + \cos 3\theta}{4}$$
$$\sin^3\theta = \frac{3 - 4\cos 2\theta + \cos 4\theta}{8} \qquad \cos^3\theta = \frac{4\cos 2\theta + \cos 4\theta}{8}$$

$$\begin{aligned}
2\cos\theta\cos\phi &= \cos(\theta-\phi) + \cos(\theta+\phi) \\
2\sin\theta\sin\phi &= \cos(\theta-\phi) - \cos(\theta+\phi) \\
2\sin\theta\cos\phi &= \sin(\theta+\phi) + \sin(\theta-\phi) \\
2\cos\theta\sin\phi &= \sin(\theta+\phi) - \sin(\theta-\phi)
\end{aligned}$$

$$\sinh z = \frac{e^z - e^{-z}}{2} \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$
$$\cosh z = \frac{e^z + e^{-z}}{2} \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$
$$\sin iz = i\sinh z \qquad \cos iz = \cosh z$$
$$i\sin z = \sinh iz \qquad \cos z = \cosh iz$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\arcsin x = \frac{1}{\sqrt{1-x^2}} \qquad \frac{\mathrm{d}}{\mathrm{d}x}\operatorname{arccsc} x = -\frac{1}{|x|\sqrt{x^2-1}}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\arccos x = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{\mathrm{d}}{\mathrm{d}x}\operatorname{arcsec} x = \frac{1}{|x|\sqrt{x^2-1}}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\arctan x = -\frac{1}{1+x^2} \qquad \frac{\mathrm{d}}{\mathrm{d}x}\operatorname{arccot} x = -\frac{1}{1-x^2}$$

3.2.1 Gram-Schmidt Orthogonalization

$$|\phi_n\rangle = |\psi_n\rangle - \sum_{i=0}^{n-1}\left\langle\hat{\phi}_i\middle|\psi_n\right\rangle\left|\hat{\phi}_i\right\rangle$$
$$\left|\hat{\phi}_n\right\rangle = \frac{|\phi_n\rangle}{\langle\phi_n|\phi_n\rangle^{\frac{1}{2}}}, \qquad \left\langle\hat{\phi}_i\middle|\hat{\phi}_j\right\rangle = \delta_{ij}$$

3.2.2 Hermitian Operators

Can be found with repeated integration by parts:

$$\int_a^b f^*(s)[\mathcal{L}g(s)]\,\mathrm{d}s = \int_a^b [\mathcal{L}f(s)]^* g(s)\,\mathrm{d}s + \underbrace{\text{boundary terms}}_{\text{reduces to 0}}$$