PC2134 AY1920 Midterm 2 Solutions

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Question 1

The Fourier transform of a function $f(x)$ is defined by:

$$\mathcal{F}\{f(x)\} = \tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x)\,\mathrm{d}x$$

provided that the integral exists.

(a) If $f(x)$ is an even function, show that:

$$\tilde{f}(k) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \cos(kx) f(x)\,\mathrm{d}x \qquad\qquad f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \cos(kx) \tilde{f}(k)\,\mathrm{d}k$$

Solution.

$$\begin{aligned}\tilde{f}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x)\,\mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \cos(kx) f(x) - i \sin(kx) f(x)\,\mathrm{d}x \\ &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \cos(kx) f(x)\,\mathrm{d}x\end{aligned}$$

We can easily check that $\tilde{f}(k)$ is also even. Hence by a similar argument the other property is easily proven.

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(b) Find the Fourier transform of $f(x) = H(a - x)H(x + a), a > 0$, where H is the Heaviside step function. Hence, evaluate the following improper integral:

$$\int_{-\infty}^{\infty} \frac{\sin(ka)\cos(kx)}{k}\,\mathrm{d}k$$

Solution.

$$\begin{aligned}\mathcal{F}\{f(x)\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} H(x + a) H(-x + a)\,\mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{-ikx}\,\mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{ik} \left(e^{ika} - e^{-ika}\right)\end{aligned}$$

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\sin(ka)\cos(kx)}{k}\,\mathrm{d}k &= \int_{-\infty}^{\infty} \frac{1}{4ik}\left[\left(e^{ika} - e^{-ika}\right)\left(e^{ikx} + e^{-ikx}\right)\right]\mathrm{d}k \\
&= \int_{-\infty}^{\infty} \frac{1}{4ik}\left[e^{ik(a+x)} - e^{-ik(a+x)} + e^{ik(a-x)} - e^{-ik(a-x)}\right]\mathrm{d}k \\
&= \frac{\sqrt{2\pi}}{4}\int_{-\infty}^{\infty} \mathcal{F}\{f(a+x)\} + \mathcal{F}\{f(a-x)\}\,\mathrm{d}k \\
&= \frac{\sqrt{2\pi}}{4}\int_{-\infty}^{\infty} e^{ikx}\mathcal{F}\{f(a+x)\} + \mathcal{F}\{f(a-x)\}\,\mathrm{d}k \\
&= \frac{\sqrt{2\pi}}{4}\left[\int_{-\infty}^{\infty} e^{ikx}\mathcal{F}\{f(x)\}\,\mathrm{d}k\,\bigg|_{x=a} + \int_{-\infty}^{\infty} e^{ikx}\mathcal{F}\{f(-x)\}\,\mathrm{d}k\,\bigg|_{x=a}\right] \\
&= \frac{\sqrt{2\pi}}{4}(f(a) + f(-a)) \\
&= 0
\end{aligned}$$

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(c) Solve the following integral equation:

$$\int_0^{\infty} f(x)\cos(kx)\,\mathrm{d}x = \begin{cases} 1-k, & 0 \le k \le 1 \\ 0, & k > 1 \end{cases}$$

Hence, or otherwise, evaluate the following improper integral:

$$\int_0^{\infty} \frac{\sin^2 x}{x^2}\,\mathrm{d}x$$

Solution. *We make an ansatz that $f(x)$ is even. Then*

$$\begin{aligned}
\int_0^{\infty} f(x)\cos(kx)\,\mathrm{d}x &= \sqrt{\frac{2}{\pi}}\tilde{f}(x)\,\mathrm{d}x \\
&= \begin{cases} 1-k, & 0 \le k \le 1 \\ 0, & k > 1 \end{cases}
\end{aligned}$$

Then taking the inverse Fourier transform on both sides,

$$\begin{aligned}
f(x) &= \sqrt{\frac{2}{\pi}}\sqrt{\frac{2}{\pi}}\int_0^{\infty} \cos(kx)\tilde{f}(k)H(1-k)H(k)\,\mathrm{d}k \\
&= \frac{2}{\pi}\int_0^1 (1-k)\cos(kx)\,\mathrm{d}x \\
&= \frac{2}{\pi}\left\{\left[\frac{\sin(kx)}{x}\right]_0^1 - \int_0^1 \frac{k}{x}\,\mathrm{d}\sin(kx)\right\} \\
&= \frac{2}{\pi}\left\{\frac{\sin x}{x} - \left[\frac{k\sin(kx)}{x^2}\right]_0^1 - \int_0^1 \frac{\sin(kx)}{x}\,\mathrm{d}k\right\} \\
&= \frac{2}{\pi}\left[\frac{1}{x^2} - \frac{\cos x}{x^2}\right]
\end{aligned}$$

Then let us evaluate the integral at $k = 0$:

$$\begin{aligned}
\int_0^\infty f(x)\cos(kx)\,\mathrm{d}x\,\bigg|_{k=0} &= \frac{2}{\pi}\left\{\int_0^\infty \frac{\cos(kx)}{x^2}\,\mathrm{d}x\,\bigg|_{k=0} - \int_0^\infty \frac{\cos^2(x)}{x^2}\,\mathrm{d}x\right\} \\
[1-k]\bigg|_{k=0} &= \frac{2}{\pi}\left\{\int_0^\infty \frac{\cos(0)-1}{x^2}\,\mathrm{d}x + \int_0^\infty \frac{\sin^2(x)}{x^2}\,\mathrm{d}x\right\} \\
\int_0^\infty \frac{\sin^2(x)}{x^2}\,\mathrm{d}x &= \frac{2}{\pi}
\end{aligned}$$

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Question 2

The Laplace transform of the function $f(t)$ is defined by:

$$\mathcal{L}\{f(t)\} = \overline{f}(s) = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$$

(a) The error function is defined as

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-\xi^2}\,\mathrm{d}\xi$$

(i) Find $\mathcal{L}\{\operatorname{erf}(\sqrt{t})\}$. Hence, or otherwise, find $\mathcal{L}\{t\operatorname{erf}(2\sqrt{t})\}$.

Solution.

<u>Method 1.</u> *This method uses integration by parts and is more of a brute force approach but is fairly fast and straightforward.*

$$\begin{aligned}
\mathcal{L}\Big\{\operatorname{erf}(\sqrt{t})\Big\} &= \int_0^\infty e^{-st}\operatorname{erf}(\sqrt{t})\,\mathrm{d}t \\
&= -\int_0^\infty \frac{1}{s}\operatorname{erf}(\sqrt{t})\,\mathrm{d}e^{-st} \\
&= -\frac{1}{s}\left\{\Big[\operatorname{erf}(\sqrt{t})e^{-st}\Big]_0^\infty - \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t}\Big(\operatorname{erf}(\sqrt{t})\Big)e^{-st}\,\mathrm{d}t\right\}
\end{aligned}$$

Noting that the Gaussian integral $\int_{-\infty}^\infty e^{-\xi^2}\,\mathrm{d}\xi = \sqrt{\pi}$, and that $\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}\sqrt{t}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}\sqrt{t}}$,

$$\begin{aligned}
\mathcal{L}\Big\{\operatorname{erf}(\sqrt{t})\Big\} &= \frac{1}{s}\left\{\frac{2}{\sqrt{\pi}}\mathcal{L}\left\{\frac{1}{2\sqrt{t}}e^{-\sqrt{t}^2}\right\}\right\} \\
&= \frac{1}{s}\left\{\frac{1}{\sqrt{\pi}}\mathcal{L}\left\{\frac{1}{\sqrt{t}}\right\}(s+1)\right\} \\
&= \frac{1}{s}\left\{\frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-st}}{\sqrt{t}}\,\mathrm{d}t\,\bigg|_{s\to s+1}\right\} \\
&= \frac{1}{s}\left\{\frac{1}{\sqrt{\pi}}\int_0^\infty \frac{2}{\sqrt{s}}e^{-\sqrt{st}^2}\,\mathrm{d}\sqrt{st}\,\bigg|_{s\to s+1}\right\} \\
&= \frac{1}{s\sqrt{s+1}}
\end{aligned}$$

METHOD 2. *Another way is through interchange of limits:*

$$
\begin{aligned}
\mathcal{L}\Big\{\operatorname{erf}(\sqrt{t})\Big\} &= \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-st}\int_0^{\sqrt{t}} e^{-\xi^2}\,\mathrm{d}\xi\,\mathrm{d}t \\
&= \frac{2}{\sqrt{\pi}}\int_{\xi=0}^\infty \int_{t=\xi^2}^\infty e^{-st}e^{-\xi^2}\,\mathrm{d}t\,\mathrm{d}\xi \\
&= \frac{2}{\sqrt{\pi}}\int_0^\infty -\frac{1}{s}\big[e^{-st}\big]_{\xi^2}^\infty e^{-\xi^2}\,\mathrm{d}\xi \\
&= \frac{2}{\sqrt{\pi}}\int_0^\infty \frac{1}{s}e^{-\xi^2(1+s)}\,\mathrm{d}\xi
\end{aligned}
$$

Taking $u = \xi\sqrt{1+s}$ *this becomes the Gaussian integral:*

$$
\begin{aligned}
\mathcal{L}\Big\{\operatorname{erf}(\sqrt{t})\Big\} &= \frac{2}{\sqrt{\pi}}\frac{1}{s\sqrt{1+s}}\int_0^\infty e^{-u^2}\,\mathrm{d}u \\
&= \frac{1}{s\sqrt{s+1}}
\end{aligned}
$$

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Then, by the scaling property of the Laplace transform we can easily see that $\mathcal{L}\big\{\operatorname{erf}(2\sqrt{t})\big\} = \frac{1}{s\sqrt{1/4+s}}$. *Hence,*

$$
\begin{aligned}
\mathcal{L}\Big\{t\operatorname{erf}(2\sqrt{t})\Big\} &= -\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}\Big\{\operatorname{erf}(2\sqrt{t})\Big\} \\
&= -\frac{\mathrm{d}}{\mathrm{d}s}\frac{2}{s\sqrt{s+4}} \\
&= \frac{3s+8}{s^2(s+4)^{\frac{3}{2}}}
\end{aligned}
$$

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(ii) Hence, evaluate the following improper integral

$$
\int_0^\infty \xi e^{-\xi^2}\operatorname{erf}(\xi)\,\mathrm{d}\xi
$$

Solution. *Substituting* $t = \xi^2$,

$$
\begin{aligned}
\int_0^\infty \xi e^{-\xi^2}\operatorname{erf}(\xi)\,\mathrm{d}\xi &= \frac{1}{2}\int_0^\infty e^{-t}\operatorname{erf}(\sqrt{t})\,\mathrm{d}u \\
&= \frac{1}{2}\mathcal{L}\Big\{\operatorname{erf}(\sqrt{t})\Big\}\Big|_{s=1} \\
&= \frac{1}{2\sqrt{2}}
\end{aligned}
$$

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(b) Find $\mathcal{L}^{-1}\Big\{\frac{s}{(s^2+a^2)^2}\Big\}$ using the Laplace convolution theorem.

Solution.

$$
\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} &= \mathcal{L}^{-1}\left\{\mathcal{L}\{\cos(at)\}\frac{1}{a}\mathcal{L}\{\sin(at)\}\right\} \\
&= \frac{1}{a}\cos(at) * \sin(at) \\
&= \frac{1}{a}\int_0^t \cos(au)\sin(at-au)\,\mathrm{d}u \\
&= \frac{1}{2a}\int_0^t \sin(at) - \sin(at-2au)\,\mathrm{d}u \\
&= \frac{1}{2a}\left\{t\sin t + \left[\frac{\cos(at-2au)}{2a}\right]_0^t\right\} \\
&= \frac{t\sin t}{2a}
\end{aligned}
$$

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Question 3

The general form of the second-order linear ordinary differential equation is given as follows:

$$
p_2(x)\frac{\mathrm{d}^2y(x)}{\mathrm{d}x^2} + p_1(x)\frac{\mathrm{d}y(x)}{\mathrm{d}x} + p_0(x)y(x) = q(x).
$$

The general solution is

$$
y(x) = c_1y_1(x) + c_2y_2(x) + y_p(x)
$$

where $y_{1,2}(x)$ are two linearly independent solutions of the homogeneous equation corresponding to $q(x) = 0, y_p(x)$ is the solution of the inhomogeneous equation and $c_{1,2}$ are two arbitrary constants. Consider the following differential equation:

$$
x^2\frac{\mathrm{d}^2y(x)}{\mathrm{d}x^2} - x\frac{\mathrm{d}y(x)}{\mathrm{d}x} + y(x) = (\ln 2)^2.
$$

(a) Show that $x = 0$ is a regular singular point.

Solution.

$$
\lim_{x\to 0}\frac{p_0(x)}{p_2(x)} = \lim_{x\to 0}\frac{1}{x^2} = \infty
$$

$$
\lim_{x\to 0}\frac{p_1(x)}{p_2(x)}\lim_{x\to 0}\frac{-x}{x^2} = -\infty
$$

Hence it is a singular point.

$$
\lim_{x\to 0}x^2\frac{p_0(x)}{p_2(x)} = \lim_{x\to 0}\frac{x^2}{x^2} = 1
$$

$$
\lim_{x\to 0}x\frac{p_1(x)}{p_2(x)}\lim_{x\to 0}\frac{-x^2}{x^2} = -1
$$

They are finite, so it is a regular singular point.

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(b) Find the roots of the indicial equation.

Solution. *We use the Frobenius method. Let* $y(x) = \sum_0^\infty a_n x^{n+\sigma}$. *Then:*

$$x^2 \sum_0^\infty a_n(n+\sigma)(n+\sigma-1)x^{n+\sigma-2} - x\sum_0^\infty a_n(n+\sigma)x^{n+\sigma-1}\sum_0^\infty a_n x^{n+\sigma} = 0$$

$$\sum_0^\infty a_n x^{n+\sigma}[(n+\sigma)(n+\sigma-1)-(n+\sigma)+1] = 0$$

The coefficients must vanish. Consider the lowest power of x : $a_0[\sigma(\sigma-1)-\sigma+1] = 0$. *Taking* $a_0 \neq 0$, *we see that* $\sigma = 1$.

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(c) Find the independent solution $y_1(x)$ of the differential equation corresponding to the larger root of the indicial equation.

Solution. *Continuing from above and taking* $\sigma = 1$,

$$\sum_0^\infty a_n x^{n+1}[n(n+1)-(n+1)+1] = 0$$

We conclude that $n = 0$. *Hence this leaves us with* $y_1(x) = a_0 x$.

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(d) Find the second independent solution $y_2(x)$ of the differential equation.

Solution. *We simply use the fact from the Wronskian that*

$$y_2(x) = y_1(x)\int^x \frac{1}{y_1^2(\xi)}\exp\left\{-\int^\xi \frac{p_1(\chi)}{p_2(\chi)}\,\mathrm{d}\chi\right\}\mathrm{d}\xi$$

. *Substituting, we get*

$$\begin{aligned} y_2(x) &= c_2 x\int^x \frac{1}{\xi^2}\exp\left\{-\int^\xi \frac{-\chi}{\chi^2}\,\mathrm{d}\chi\right\}\mathrm{d}\xi \\ &= c_2 x\int^x \frac{1}{\xi^2}\xi\,\mathrm{d}\xi \\ &= c_2 x\ln x \end{aligned}$$

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(e) Find the general solution of the inhomogeneous differential equation.

Solution. *Variation of parameters:*

$$\begin{aligned}
y_p(x) &= -\int^x \frac{q(\xi)}{p_2(\xi)} \frac{y_2(\xi)y_1(x) - y_1(\xi)y_2(x)}{y_1(\xi)y_2'(\xi) - y_2(\xi)y_1'(\xi)} \, \mathrm{d}\xi \\
&= -\int^x \frac{\ln^2 \xi}{\xi^2} \frac{\xi x \ln \xi - \xi x \ln x}{\xi(1 + \ln \xi) - \xi \ln \xi} \, \mathrm{d}\xi \\
&= -\int^x \frac{\ln^2 \xi}{\xi^2} (x \ln \xi - x \ln x) \, \mathrm{d}\xi \\
&= \ln^2 x + 4 \ln x + 6
\end{aligned}$$

Thus $y_p(x) = c_1 x + c_2 x \ln x + \ln^2 x + 4 \ln x + 6$

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